Complex Analysis

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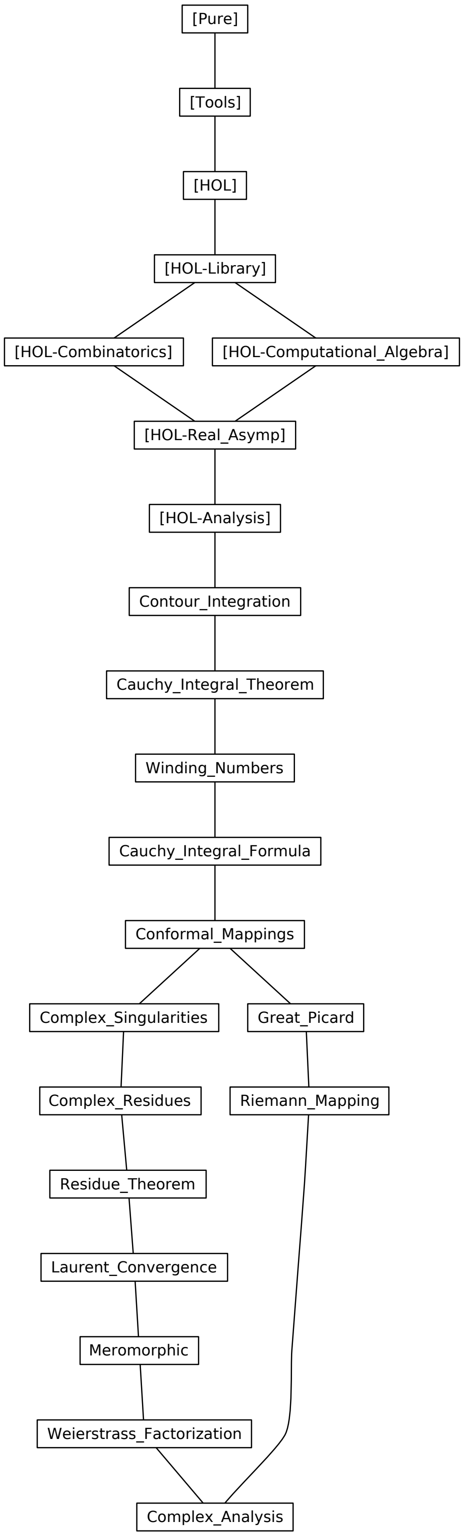
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** ‹*has'_contour'_integral*› *50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0..1*}))
has_integral i) {*0..1*}

definition *contour_integrable_on*
(**infixr** ‹*contour'_integrable'_on*› *50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i=0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f′ x*) (*at x within S*)
and *valid_path g* *path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f′ has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative h* (*at t*))

shows *contour_integral g* (λw. *contour_integral h* (*f w*)) =
contour_integral h (λz. *contour_integral g* (λw. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λx. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes $s \leq t$
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. $s \leq x \wedge x \leq t$}

corollary *contour_integral_bound_part_circlepath_strong*:
assumes *f contour_integrable_on part_circlepath z r s t*
and *finite k* **and** $0 \leq B$ $0 < r$ $s \leq t$
and ⋀*x*. *x* ∈ *path_image*(*part_circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod* (*contour_integral* (*part_circlepath z r s t*) *f*) ≤ *B* ∗ *r* ∗ (*t* − *s*)

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λn::'*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0*..*1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λn. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* ⇒ *complex*, *complex*, *real*, *real* ⇒ *complex*, *complex*] ⇒ *bool* **where**
winding_number_prop γ *z e p n* ≡

valid_path p ∧ *z* ∉ *path_image p* ∧
pathstart p = *pathstart* γ ∧
pathfinish p = *pathfinish* γ ∧
(∀ *t* ∈ {*0..1*}. *norm*(γ *t* − *p t*) < *e*) ∧
contour_integral p (λ*w*. *1*/(*w* − *z*)) = *2* ∗ *pi* ∗ i ∗ *n*

definition *winding_number*:: [*real* ⇒ *complex*, *complex*] ⇒ *complex* **where**
winding_number γ *z* ≡ *SOME n*. ∀ *e* > *0*. ∃ *p*. *winding_number_prop* γ *z e p n*

proposition *winding_number_valid_path*:
assumes *valid_path* γ *z* ∉ *path_image* γ
shows *winding_number* γ *z* = *1*/(*2*∗*pi*∗i) ∗ *contour_integral* γ (λ*w*. *1*/(*w* − *z*))

proposition *has_contour_integral_winding_number*:
assumes γ: *valid_path* γ *z* ∉ *path_image* γ
shows ((λ*w*. *1*/(*w* − *z*)) *has_contour_integral* (*2*∗*pi*∗i∗*winding_number* γ *z*)) γ

3.2 The winding number is an integer

theorem *integer_winding_number*:
⟦*path* γ; *pathfinish* γ = *pathstart* γ; *z* ∉ *path_image* γ⟧ ⟹ *winding_number* γ *z* ∈ ℤ

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes *z*::*complex*
assumes γ: *path* γ **and** *z*: *z* ∉ *path_image* γ
shows *continuous* (*at z*) (*winding_number* γ)

corollary *continuous_on_winding_number*:
path γ ⟹ *continuous_on* (− *path_image* γ) (λ*w*. *winding_number* γ *w*)

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** *z*: *z* ∈ *outside* (*path_image* γ)
shows *winding_number* γ *z* = *0*

proposition *winding_number_part_circlepath_pos_less*:
assumes *s* < *t* **and** *no*: *norm*(*w* − *z*) < *r*

shows $0 < Re\ (winding\_number(part\_circlepath\ z\ r\ s\ t)\ w)$

proposition *winding_number_circlepath*:
assumes $norm(w - z) < r$ **shows** $winding\_number(circlepath\ z\ r)\ w = 1$

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes z: $z \in interior(convex\ hull\ \{a,b,c\})$
shows $winding\_number(linepath\ a\ b\ {+}{+}{+}\ linepath\ b\ c\ {+}{+}{+}\ linepath\ c\ a)\ z =$
$(if\ 0 < Im((b - a) * cnj\ (b - z))\ then\ 1\ else\ -1)$

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes $simple\_path\ \gamma$
obtains $\bigwedge z.\ z \in inside(path\_image\ \gamma) \Longrightarrow winding\_number\ \gamma\ z = 1$
$\mid \bigwedge z.\ z \in inside(path\_image\ \gamma) \Longrightarrow winding\_number\ \gamma\ z = -1$

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in box\ a1\ a3$
shows $winding\_number\ (rectpath\ a1\ a3)\ z = 1$

proposition *winding_number_rectpath_outside*:
assumes $Re\ a1 \le Re\ a3\ Im\ a1 \le Im\ a3$
assumes $z \notin cbox\ a1\ a3$
shows $winding\_number\ (rectpath\ a1\ a3)\ z = 0$

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** $z \in interior\ S\ valid\_path\ \gamma\ path\_image\ \gamma \subseteq S - \{z\}$

pathfinish γ = *pathstart* γ
shows ((λw. *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ
theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: *norm*(*w*−*z*) < *r*
shows ((λu. *f u*/(*u*−*w*)) *has_contour_integral* (*2* ∗ *of_real pi* ∗ i ∗ *f w*)) (*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: $\bigwedge$*z*. *z* ∈ *S* ⟹ (*f has_field_derivative f′ z*) (*at z*)
shows *f′ holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
⟦*continuous_on S f*; *open S*;
$\bigwedge$*a b c*. *convex hull* {*a*,*b*,*c*} ⊆ *S*
⟶ *contour_integral* (*linepath a b*) *f* +
contour_integral (*linepath b c*) *f* +
contour_integral (*linepath c a*) *f* = *0*⟧
⟹ *f analytic_on S*

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* (*S*−*K*) **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: ($\bigwedge$*x*. *x* ∈ *interior S* − *K* ⟹ *f field_differentiable at x*)
and *z*: *z* ∈ *interior S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* γ ⊆ *S* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
shows ((λw. *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* (*cball z r*) *f f holomorphic_on ball z r w* ∈ *ball z r*

shows *contour_integral*(*circlepath z r*) ($\lambda u.$ *f u*/(*u*−*w*)^(*Suc k*)) = (*2* ∗ *pi* ∗ i) ∗ (*deriv* ^^ *k*) *f w* / (*fact k*)

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: *f holomorphic_on ball z r*
and *w*: *w* ∈ *ball z r*
shows (($\lambda n.$ (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w*−*z*)^*n*) *sums f w*)

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes *f holomorphic_on UNIV* **and** (*f* ⟶ *l*) *at_infinity*
shows *f z* = *l*

proposition *Liouville_weak_inverse*:
assumes *f holomorphic_on UNIV* **and** *unbounded*: ⋀*B*. *eventually* ($\lambda x.$ *norm* (*f x*) ≥ *B*) *at_infinity*
obtains *z* **where** *f z* = *0*
theorem *fundamental_theorem_of_algebra*:
fixes *a* :: *nat* ⇒ *complex*
assumes *a 0* = *0* ∨ (∃ *i* ∈ {*1..n*}. *a i* ≠ *0*)
obtains *z* **where** (∑ *i*≤*n*. *a i* ∗ *z*^*i*) = *0*

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes *z*::*complex*
assumes *cont*: *eventually* ($\lambda n.$ *continuous_on* (*cball z r*) (*f n*) ∧
(∀ *w* ∈ *ball z r*. ((*f n*) *has_field_derivative* (*f′ n w*)) (*at w*))) *F*
and *ulim*: *uniform_limit* (*cball z r*) *f g F*
and *F*: ¬ *trivial_limit F* **and** *0* < *r*
obtains *g′* **where**
continuous_on (*cball z r*) *g*
⋀*w*. *w* ∈ *ball z r* ⟹ (*g has_field_derivative* (*g′ w*)) (*at w*) ∧ (($\lambda n.$ *f′ n w*) ⟶ *g′ w*) *F*

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:

f holomorphic_on ball z r ⟷
(∀ *w* ∈ *ball z r*. (λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w*−*z*)^*n*) *sums f w*)

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *z*: *z* ∈ *S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* γ ⊆ *S* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: ⋀*w*. *w* ∉ *S* ⟹ *winding_number* γ *w* = *0*
shows ((λ*w*. *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* γ ⊆ *S*
and *zero*: ⋀*w*. *w* ∉ *S* ⟹ *winding_number* γ *w* = *0*
shows (*f has_contour_integral 0*) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* γ ⊆ *S*
⋀*w*. *w* ∉ *S* ⟹ *w* ∈ *outside*(*path_image* γ)
shows (*f has_contour_integral 0*) γ

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded* (*range f*)
shows *f constant_on UNIV*

4.11 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps*
where
fps_expansion f z0 = *Abs_fps* (λ*n*. (*deriv* ^^ *n*) *f z0* / *fact n*)

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*

imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
  assumes *holf*: f *holomorphic_on* S
    and *open* S *connected* S $\xi \in S$ $f\ \xi = 0$ $\beta \in S$ $f\ \beta \neq 0$
   obtains r **where** $0 < r$ **and** *ball* $\xi\ r \subseteq S$ **and**
     $\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
  assumes *holf*: f *holomorphic_on* S
    and *open* S **and** *connected* S
    and $U \subseteq S$ **and** $\xi \in S$
    and ξ *islimpt* U
    and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
    and $w \in S$
   shows $f\ w = 0$

corollary *analytic_continuation_open*:
  assumes *open* s **and** *open* s' **and** $s \neq \{\}$ **and** *connected* s'
    and $s \subseteq s'$
  assumes f *holomorphic_on* s' **and** g *holomorphic_on* s'
    and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
  assumes $z \in s'$
  shows $f\ z = g\ z$

corollary *analytic_continuation'*:
  assumes f *holomorphic_on* S *open* S *connected* S
    and $U \subseteq S$ $\xi \in S$ ξ *islimpt* U
    and f *constant_on* U
   shows f *constant_on* S

5.2 Open mapping theorem

theorem *open_mapping_thm*:
  assumes *holf*: f *holomorphic_on* S
    and S: *open* S **and** *connected* S
    and *open* U **and** $U \subseteq S$
    and *fne*: $\neg$ f *constant_on* S
   shows *open* $(f\ `\ U)$

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:

assumes *holf*: *f holomorphic_on S*
and *S*: *open S* **and** *connected S*
and *open U* **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \le norm(f\ \xi)$
shows *f constant_on S*

proposition *maximum_modulus_frontier*:
assumes *holf*: *f holomorphic_on (interior S)*
and *contf*: *continuous_on (closure S) f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow norm(f\ z) \le B$
and $\xi \in S$
shows $norm(f\ \xi) \le B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on (f ‘ S)*
$\bigwedge z.\ z \in S \Longrightarrow deriv\ f\ z * deriv\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on (ball 0 1)* **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
and ξ: $norm\ \xi < 1$
shows $norm\ (f\ \xi) \le norm\ \xi$ **and** $norm(deriv\ f\ 0) \le 1$
and $((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1$
(**is** $?P \Longrightarrow ?Q$)

corollary *Schwarz_Lemma'*:
assumes *holf*: *f holomorphic_on (ball 0 1)* **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
shows $((\forall \xi.\ norm\ \xi < 1 \longrightarrow norm\ (f\ \xi) \le norm\ \xi)$
$\wedge\ norm(deriv\ f\ 0) \le 1)$
$\wedge\ (((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
  assumes *open S* **and** *cnjs*: *cnj ‘ S* $\subseteq$ *S*
    and *holf*: *f holomorphic_on* ($S \cap \{z.\ 0 < Im\ z\}$)
    and *contf*: *continuous_on* ($S \cap \{z.\ 0 \le Im\ z\}$) *f*
    and *f*: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
   shows ($\lambda z.$ *if* $0 \le Im\ z$ *then f z else cnj*(*f*(*cnj z*))) *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
  assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a* = *1*
  obtains *b r* **where** $1/12 < r$ **and** *ball b r* $\subseteq$ *f ‘* (*ball a 1*)

theorem *Bloch*:
  assumes *holf*: *f holomorphic_on ball a r* **and** $0 < r$
    and r': $r' \le r * norm\ (deriv\ f\ a)\ /\ 12$
  obtains *b* **where** *ball b r′* $\subseteq$ *f ‘* (*ball a r*)

corollary *Bloch_general*:
  assumes *holf*: *f holomorphic_on S* **and** $a \in S$
    and *tle*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow t \le dist\ a\ z$
    and *rle*: $r \le t * norm(deriv\ f\ a)\ /\ 12$
  obtains *b* **where** *ball b r* $\subseteq$ *f ‘ S*

end

6 The Great Picard Theorem and its Applications

theory *Great_Picard*
  imports *Conformal_Mappings*
begin

6.1 Schottky's theorem

theorem *Schottky*:
  assumes *holf*: *f holomorphic_on cball 0 1*
    and *nof0*: *norm*(*f 0*) $\le r$

and *not01*: $\bigwedge z.\ z \in cball\ 0\ 1 \Longrightarrow \neg(f\ z = 0 \lor f\ z = 1)$
and $0 < t\ t < 1\ norm\ z \le t$
shows $norm(f\ z) \le exp(pi * exp(pi * (2 + 2 * r + 12 * t\ /\ (1 - t))))$

6.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains *R*
where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f\ holomorphic\_on\ cball\ 0\ (R(f\ 0));$
$\bigwedge z.\ norm\ z \le R(f\ 0) \Longrightarrow f\ z \ne 0 \land f\ z \ne 1]\!] \Longrightarrow norm(deriv\ f\ 0)$
< 1

theorem *little_Picard*:
assumes *holf*: *f holomorphic_on UNIV*
and $a \ne b\ range\ f \cap \{a,b\} = \{\}$
obtains *c* **where** $f = (\lambda x.\ c)$

6.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F} :: [nat,'a::euclidean\_space] \Rightarrow 'b::\{real\_normed\_vector,heine\_borel\}$
assumes *compact S*
and *M*: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \le M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!]$
$\Longrightarrow \exists d.\ 0 < d \land (\forall n\ y.\ y \in S \land norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n$
$x - \mathcal{F}\ n\ y) < e)$
obtains *g k* **where** *continuous_on S g strict_mono* $(k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \ge N \land x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x -$
$g\ x) < e$

6.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F} :: [nat,complex] \Rightarrow complex$
assumes *open S*
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h\ holomorphic\_on\ S$
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists B.\ \forall h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h$
$z) \le B$
and *rng_f*: $range\ \mathcal{F} \subseteq \mathcal{H}$
obtains *g r*
where *g holomorphic_on S strict_mono* $(r :: nat \Rightarrow nat)$

$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ (\mathcal{F} \circ r)\ g$ *sequentially*

6.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
and $z0 \in S$
shows $g\ z0 \neq 0$

corollary *Hurwitz_injective*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *inj*: $\bigwedge n.\ inj\_on\ (\mathcal{F}\ n)\ S$
shows *inj_on g S*

6.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open M* $z \in M$ $a \neq b$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
obtains *l* **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)$ $\bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains *a* **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)$ $\bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$

obtains a **where** $\bigwedge w.\ w \neq a \Longrightarrow$ *infinite* $\{x.\ x \in M - \{z\} \wedge f\ x = w\}$

theorem *Casorati_Weierstrass*:
assumes *open M* $z \in M$ *f holomorphic_on* $(M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)$ $\bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
shows *closure*$(f\ `\ (M - \{z\})) = UNIV$

end

7 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

7.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real*,*complex*,*complex*] ⇒ *complex* **where**
Moebius_function ≡ $\lambda t\ w\ z.\ exp(\mathrm{i} * of\_real\ t) * (z - w) / (1 - cnj\ w * z)$

7.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g z*. *path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ (*z* ∉ *S*)
⟶ *winding_number g z* = *0*) (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g f*. *valid_path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ *f holomorphic_on S*
⟶ (*f has_contour_integral 0*) *g*) (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ⟶
(∃ *h*. ∀ *z*. *z* ∈ *S* ⟶ (*h has_field_derivative f z*) (*at z*))) (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S ⟷

connected $S \wedge$
$(\forall f.$ *f holomorphic_on* $S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.$ *g holomorphic_on* $S \wedge (\forall z \in S.\ f\ z = exp(g\ z))))$ (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
simply_connected $S \longleftrightarrow$
connected $S \wedge$
$(\forall f.$ *f holomorphic_on* $S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.$ *g holomorphic_on* $S \wedge (\forall z \in S.\ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected $S \longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.$ *f holomorphic_on* $S \wedge$ *g holomorphic_on ball 0 1* $\wedge$
$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$ (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected $S \longleftrightarrow S = \{\} \vee S$ *homeomorphic ball* $(0::complex)\ 1$
(**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes S :: *complex set*
assumes *open S*
shows *contractible* $S \longleftrightarrow$ *simply_connected S*

7.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes S :: *complex set*
assumes *open S*
shows *simply_connected_eq_frontier_properties*:
simply_connected $S \longleftrightarrow$
connected $S \wedge$
(*if bounded S then connected*(*frontier S*)
else $(\forall C \in$ *components*(*frontier S*). $\neg$*bounded C*)) (**is** *?fp*)
and *simply_connected_eq_unbounded_complement_components*:
simply_connected $S \longleftrightarrow$
connected $S \wedge (\forall C \in$ *components*$(-\ S).\ \neg$*bounded C*) (**is** *?ucc*)
and *simply_connected_eq_empty_inside*:
simply_connected $S \longleftrightarrow$
connected $S \wedge$ *inside* $S = \{\}$ (**is** *?ei*)

7.4 Further equivalences based on continuous logs and sqrts

proposition

fixes S :: *complex set*
assumes *open* S
shows *simply_connected_eq_continuous_log*:
simply_connected $S \longleftrightarrow$
connected S $\wedge$
$(\forall f$::*complex*⇒*complex*. *continuous_on* S f $\wedge$ $(\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow$ $(\exists g.$ *continuous_on* S g $\wedge$ $(\forall z \in S.\ f\ z = exp\ (g\ z))))$ (**is** *?log*)
and *simply_connected_eq_continuous_sqrt*:
simply_connected $S \longleftrightarrow$
connected S $\wedge$
$(\forall f$::*complex*⇒*complex*. *continuous_on* S f $\wedge$ $(\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow$ $(\exists g.$ *continuous_on* S g $\wedge$ $(\forall z \in S.\ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)

7.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
open S $\wedge$ *simply_connected* $S \longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f$ *holomorphic_on* S $\wedge$ g *holomorphic_on ball 0 1* $\wedge$
$(\forall z \in S.\ f\ z \in$ *ball 0 1* $\wedge\ g(f\ z) = z)$ $\wedge$
$(\forall z \in$ *ball 0 1*. $g\ z \in S \wedge f(g\ z) = z))$
(**is** _ = *?rhs*)

7.6 Applications to Winding Numbers

7.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path p* **and** ζp: $\zeta \notin$ *path_image p*
and *path q* **and** ζq: $\zeta \notin$ *path_image q*
and *qp*: *pathstart q = pathstart p pathfinish q = pathfinish p*
shows *winding_number p* ζ = *winding_number q* ζ $\longleftrightarrow$ *homotopic_paths* $(-\{\zeta\})$ *p q*
(**is** *?lhs* = *?rhs*)

end
theory *Complex_Singularities*
imports *Conformal_Mappings*
begin

7.8 Non-essential singular points

definition
is_pole :: ('*a*::*topological_space* ⇒ '*b*::*real_normed_vector*) ⇒ '*a* ⇒ *bool*

where *is_pole f a* = (*LIM x* (*at a*). *f x* :> *at_infinity*)

7.9 Isolated singularities

7.10 The order of non-essential singularities (i.e. removable singularities or poles)

definition *zorder* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *int* **where**
zorder f z = (*THE n*. (∃ *h r*. *r*>*0* ∧ *h holomorphic_on cball z r* ∧ *h z*≠*0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi n*
∧ *h w* ≠*0*)))

definition *zor_poly*
:: [*complex* ⇒ *complex*, *complex*] ⇒ *complex* ⇒ *complex* **where**
zor_poly f z = (*SOME h*. ∃ *r*. *r* > *0* ∧ *h holomorphic_on cball z r* ∧ *h z* ≠ *0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi* (*zorder f z*)
∧ *h w* ≠*0*))

7.11 Isolated points

7.12 Isolated zeros

end
theory *Complex_Residues*
imports *Complex_Singularities*
begin

7.13 Definition of residues

definition *residue* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
residue f z = (*SOME int*. ∃ *e*>*0*. ∀ ε>*0*. ε<*e*
⟶ (*f has_contour_integral 2*∗*pi*∗ i ∗*int*) (*circlepath z* ε))

theorem *residue_fps_expansion_over_power_at_0*:
assumes *f has_fps_expansion F*
shows *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *0* = *fps_nth F n*

7.14 Poles and residues of some well-known functions

end

8 The Residue Theorem, the Argument Principle and Rouché's Theorem

theory *Residue_Theorem*
imports *Complex_Residues HOL−Library.Landau_Symbols*
begin

8.1 Cauchy's residue theorem

theorem *Residue_theorem*:
fixes *S pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
and *g*::*real* ⇒ *complex*
assumes *open S connected S finite pts* **and**
holo:*f holomorphic_on S*−*pts* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_image g ⊆ *S*−*pts* **and**
homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0*
shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

8.2 The argument principle

theorem *argument_principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles S*:: *complex set*
defines *pz* ≡ {*w*∈*S*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
assumes *open S connected S* **and**
f_holo:*f holomorphic_on S*−*poles* **and**
h_holo:*h holomorphic_on S* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_img:*path_image g* ⊆ *S* − *pz* **and**
homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0* **and**
finite:*finite pz* **and**
poles:∀ *p*∈*S*∩*poles*. *is_pole f p*
shows *contour_integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
(∑ *p*∈*pz*. *winding_number g p* ∗ *h p* ∗ *zorder f p*)
(**is** *?L*=*?R*)

8.3 Coefficient asymptotics for generating functions

theorem
fixes *f* :: *complex* ⇒ *complex* **and** *n* :: *nat* **and** *r* :: *real*
defines *g* ≡ (λ*w*. *f w* / *w* ^ *Suc n*) **and** γ ≡ *circlepath 0 r*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*

shows *fps_coeff_conv_residues*:
$(deriv\ \widehat{}\widehat{}\ n)\ f\ 0\ /\ fact\ n =$
$contour\_integral\ \gamma\ g\ /\ (2 * pi * \mathrm{i}) - (\sum z{\in}S.\ residue\ g\ z)$ (**is** *?thesis1*)
and *fps_coeff_residues_bound*:
$(\bigwedge z.\ norm\ z = r \Longrightarrow z \notin k \Longrightarrow norm\ (f\ z) \leq C) \Longrightarrow C \geq 0 \Longrightarrow finite\ k \Longrightarrow$
$norm\ ((deriv\ \widehat{}\widehat{}\ n)\ f\ 0\ /\ fact\ n + (\sum z{\in}S.\ residue\ g\ z)) \leq C\ /\ r\ \widehat{}\ n$

corollary *fps_coeff_residues_bigo*:
fixes $f :: complex \Rightarrow complex$ **and** $r :: real$
assumes $open\ A\ connected\ A\ cball\ 0\ r \subseteq A\ r > 0$
assumes $f\ holomorphic\_on\ A - S\ S \subseteq ball\ 0\ r\ finite\ S\ 0 \notin S$
assumes g: $eventually\ (\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \widehat{}\ Suc\ n)\ z))\ sequentially$
(**is** $eventually\ (\lambda n.\ \_ = -?g'\ n)\ \_$)
shows $(\lambda n.\ (deriv\ \widehat{}\widehat{}\ n)\ f\ 0\ /\ fact\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \widehat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - \_) \in O(\_)$)

corollary *fps_coeff_residues_bigo'*:
fixes $f :: complex \Rightarrow complex$ **and** $r :: real$
assumes *exp*: $f\ has\_fps\_expansion\ F$
assumes $open\ A\ connected\ A\ cball\ 0\ r \subseteq A\ r > 0$
assumes $f\ holomorphic\_on\ A - S\ S \subseteq ball\ 0\ r\ finite\ S\ 0 \notin S$
assumes $eventually\ (\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \widehat{}\ Suc\ n)\ z))\ sequentially$
(**is** $eventually\ (\lambda n.\ \_ = -?g'\ n)\ \_$)
shows $(\lambda n.\ fps\_nth\ F\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \widehat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - \_) \in O(\_)$)

8.4 Rouche's theorem

theorem *Rouche_theorem*:
fixes $f\ g::complex \Rightarrow complex$ **and** $s::\ complex\ set$
defines $fg \equiv (\lambda p.\ f\ p + g\ p)$
defines $zeros\_fg \equiv \{p{\in}s.\ fg\ p = 0\}$ **and** $zeros\_f \equiv \{p{\in}s.\ f\ p = 0\}$
assumes
open s **and** *connected s* **and**
finite zeros_fg **and**
finite zeros_f **and**
f_holo:*f holomorphic_on s* **and**
g_holo:*g holomorphic_on s* **and**
$valid\_path\ \gamma$ **and**
loop:$pathfinish\ \gamma = pathstart\ \gamma$ **and**
path_img:$path\_image\ \gamma \subseteq s$ **and**
path_less:$\forall z{\in}path\_image\ \gamma.\ cmod(f\ z) > cmod(g\ z)$ **and**
homo:$\forall z.\ (z \notin s) \longrightarrow winding\_number\ \gamma\ z = 0$
shows $(\sum p{\in}zeros\_fg.\ winding\_number\ \gamma\ p * zorder\ fg\ p)$
$= (\sum p{\in}zeros\_f.\ winding\_number\ \gamma\ p * zorder\ f\ p)$

end
theory *Laurent_Convergence*
imports *HOL−Computational_Algebra.Formal_Laurent_Series HOL−Library.Landau_Symbols Residue_Theorem*

begin

definition *fls_conv_radius* :: *complex fls* ⇒ *ereal* **where**
fls_conv_radius f = *fps_conv_radius* (*fls_regpart f*)

definition *eval_fls* :: *complex fls* ⇒ *complex* ⇒ *complex* **where**
eval_fls F z = *eval_fps* (*fls_base_factor_to_fps F*) *z* ∗ *z powi fls_subdegree F*

definition
has_laurent_expansion :: (*complex* ⇒ *complex*) ⇒ *complex fls* ⇒ *bool*
(**infixl** ‹*has′_laurent′_expansion*› *60*)
where (*f has_laurent_expansion F*) ⟷
fls_conv_radius F > *0* ∧ *eventually* (λ*z*. *eval_fls F z* = *f z*) (*at 0*)

theorem *sums_eval_fls*:
fixes *f*
defines *n* ≡ *fls_subdegree f*
assumes *norm z* < *fls_conv_radius f* **and** *z* ≠ *0* ∨ *n* ≥ *0*
shows (λ*k*. *fls_nth f* (*int k* + *n*) ∗ *z powi* (*int k* + *n*)) *sums eval_fls f z*

theorem *not_essential_has_laurent_expansion_0*:
assumes *isolated_singularity_at f 0 not_essential f 0*
shows *f has_laurent_expansion laurent_expansion f 0*

8.5 More Laurent expansions

8.6 Formal convergence versus analytic convergence

proposition *uniform_limit_imp_fps_expansion_eq*:
fixes *f* :: *′a* ⇒ *complex fps*
assumes *lim1*: (*f* ⟶ *h*) *F*
assumes *lim2*: *uniform_limit A* (λ*x z*. *f′ x z*) *g′ F*
assumes *expansions*: *eventually* (λ*x*. *f′ x has_fps_expansion f x*) *F g′ has_fps_expansion g*
assumes *holo*: *eventually* (λ*x*. *f′ x holomorphic_on A*) *F*
assumes *A*: *open A 0* ∈ *A*

assumes *nontriv* [*simp*]: $F \neq bot$
shows $g = h$

end

theory *Meromorphic* **imports**
Laurent_Convergence
Cauchy_Integral_Formula
HOL−Analysis.Sparse_In
begin

8.7 Remove singular points

definition *remove_sings* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
remove_sings f z = (*if* ∃ *c*. *f* −*z*→ *c then Lim* (*at z*) *f else 0*)

8.8 Meromorphicity

definition *meromorphic_on* :: (*complex* ⇒ *complex*) ⇒ *complex set* ⇒ *bool*
(**infixl** ‹(*meromorphic′_on*)› *50*) **where**
f meromorphic_on A ⟷ (∀ *z*∈*A*. ∃ *F*. (λ*w*. *f* (*z* + *w*)) *has_laurent_expansion F*)

8.9 Nice meromorphicity

8.10 Closure properties and proofs for individual functions

8.11 Meromorphic functions and zorder

8.12 More on poles and zeros

end

9 The Weierstraß Factorisation Theorem

theory *Weierstrass_Factorization*
imports *Meromorphic*
begin

9.1 The elementary factors

9.2 Infinite products of elementary factors

9.3 Writing a quotient as an exponential

9.4 Constructing the sequence of zeros

9.5 The factorisation theorem for holomorphic functions

theorem *weierstrass_factorization*:
  assumes *g holomorphic_on A open A connected A*
  assumes $\bigwedge z.\ z \in frontier\ A \Longrightarrow \neg z\ islimpt\ \{w{\in}A.\ g\ w = 0\}$
  obtains *h f* **where**
    h holomorphic_on A f holomorphic_on UNIV
    $\forall z.\ f\ z = 0 \longleftrightarrow (\forall z{\in}A.\ g\ z = 0) \vee (z \in A \wedge g\ z = 0)$
    $\forall z{\in}A.\ zorder\ f\ z = zorder\ g\ z$
    $\forall z{\in}A.\ h\ z \neq 0$
    $\forall z{\in}A.\ g\ z = h\ z * f\ z$
theorem *weierstrass_factorization_UNIV*:
  assumes *g holomorphic_on UNIV*
  obtains *h f* **where**
    h holomorphic_on UNIV f holomorphic_on UNIV
    $\forall z.\ f\ z = 0 \longleftrightarrow g\ z = 0$
    $\forall z.\ zorder\ f\ z = zorder\ g\ z$
    $\forall z.\ h\ z \neq 0$
    $\forall z.\ g\ z = h\ z * f\ z$

9.6 The factorisation theorem for meromorphic functions

theorem *weierstrass_factorization_meromorphic*:
  assumes *mero*: *g nicely_meromorphic_on A* **and** *A*: *open A connected A*
  assumes *no_limpt*: $\bigwedge z.\ z \in frontier\ A \Longrightarrow \neg z\ islimpt\ \{w{\in}A.\ g\ w = 0 \vee is\_pole\ g\ w\}$
  obtains *h f1 f2* **where**
    h holomorphic_on A f1 holomorphic_on UNIV f2 holomorphic_on UNIV
    $\forall z{\in}A.\ f1\ z = 0 \longleftrightarrow \neg is\_pole\ g\ z \wedge g\ z = 0$
    $\forall z{\in}A.\ f2\ z = 0 \longleftrightarrow is\_pole\ g\ z$
    $\forall z{\in}A.\ \neg is\_pole\ g\ z \longrightarrow zorder\ f1\ z = zorder\ g\ z$
    $\forall z{\in}A.\ is\_pole\ g\ z \longrightarrow zorder\ f2\ z = -zorder\ g\ z$
    $\forall z{\in}A.\ h\ z \neq 0$
    $\forall z{\in}A.\ g\ z = h\ z * f1\ z\ /\ f2\ z$
theorem *weierstrass_factorization_meromorphic_UNIV*:
  assumes *g nicely_meromorphic_on UNIV*
  obtains *h f1 f2* **where**
    h holomorphic_on UNIV f1 holomorphic_on UNIV f2 holomorphic_on UNIV

$\forall z.\ f1\ z = 0 \longleftrightarrow \neg is\_pole\ g\ z \wedge g\ z = 0$
$\forall z.\ f2\ z = 0 \longleftrightarrow is\_pole\ g\ z$
$\forall z.\ \neg is\_pole\ g\ z \longrightarrow zorder\ f1\ z = zorder\ g\ z$
$\forall z.\ is\_pole\ g\ z \longrightarrow zorder\ f2\ z = -zorder\ g\ z$
$\forall z.\ h\ z \neq 0$
$\forall z.\ g\ z = h\ z * f1\ z\ /\ f2\ z$

end
theory *Complex_Analysis*
imports
Riemann_Mapping
Residue_Theorem
Weierstrass_Factorization
begin

end

References

[1]