The Hahn-Banach Theorem for Real Vector Spaces

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December 17, 2025

Abstract

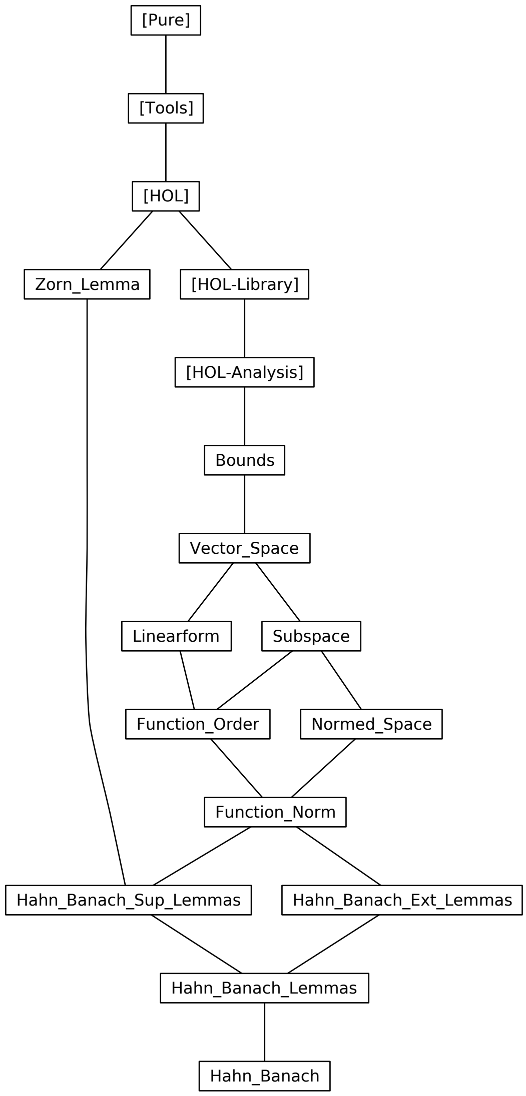
The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

Contents

1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser's textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.


[Pure]
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Bounds
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Part I
Basic Notions

2 Bounds

theory *Bounds*
imports *Main HOL−Analysis.Continuum-Not-Denumerable*
begin

locale *lub* =
  fixes A **and** x
  assumes *least* [*intro?*]: $(\bigwedge a.\ a \in A \Longrightarrow a \leq b) \Longrightarrow x \leq b$
    and *upper* [*intro?*]: $a \in A \Longrightarrow a \leq x$

lemmas [*elim?*] = *lub.least lub.upper*

definition *the-lub* :: *'a::order set* ⇒ *'a* (‹⨆-› [*90*] *90*)
  where *the-lub A* = *The* (*lub A*)

lemma *the-lub-equality* [*elim?*]:
  assumes *lub A x*
  shows $\bigsqcup A = (x{::}'a{::}order)$
⟨*proof*⟩

lemma *the-lubI-ex*:
  assumes *ex*: $\exists x.\ lub\ A\ x$
  shows *lub A* ($\bigsqcup A$)
⟨*proof*⟩

lemma *real-complete*: $\exists a{::}real.\ a \in A \Longrightarrow \exists y.\ \forall a \in A.\ a \leq y \Longrightarrow \exists x.\ lub\ A\ x$
  ⟨*proof*⟩

end

3 Vector spaces

theory *Vector-Space*
imports *Complex-Main Bounds*
begin

3.1 Signature

For the definition of real vector spaces a type $'a$ of the sort {*plus*, *minus*, *zero*} is considered, on which a real scalar multiplication · is declared.

consts
  prod :: *real* ⇒ *'a::{plus,minus,zero}* ⇒ *'a* (**infixr** ‹·› *70*)

3.2 Vector space laws

A *vector space* is a non-empty set V of elements from $'a$ with the following vector space laws: The set V is closed under addition and scalar multiplication, addition is associative and commutative; $- x$ is the inverse of x wrt. addition and 0 is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number 1 is the neutral element of scalar multiplication.

locale *vectorspace* =
  fixes V
  assumes *non-empty* [*iff*, *intro?*]: $V \neq \{\}$
    and *add-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$
    and *mult-closed* [*iff*]: $x \in V \Longrightarrow a \cdot x \in V$
    and *add-assoc*: $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$
    and *add-commute*: $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$
    and *diff-self* [*simp*]: $x \in V \Longrightarrow x - x = 0$
    and *add-zero-left* [*simp*]: $x \in V \Longrightarrow 0 + x = x$
    and *add-mult-distrib1*: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$
    and *add-mult-distrib2*: $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$
    and *mult-assoc*: $x \in V \Longrightarrow (a * b) \cdot x = a \cdot (b \cdot x)$
    and *mult-1* [*simp*]: $x \in V \Longrightarrow 1 \cdot x = x$
    and *negate-eq1*: $x \in V \Longrightarrow - x = (- 1) \cdot x$
    and *diff-eq1*: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + - y$
begin

lemma *negate-eq2*: $x \in V \Longrightarrow (- 1) \cdot x = - x$
  ⟨*proof*⟩

lemma *negate-eq2a*: $x \in V \Longrightarrow -1 \cdot x = - x$
  ⟨*proof*⟩

lemma *diff-eq2*: $x \in V \Longrightarrow y \in V \Longrightarrow x + - y = x - y$
  ⟨*proof*⟩

lemma *diff-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$
  ⟨*proof*⟩

lemma *neg-closed* [*iff*]: $x \in V \Longrightarrow - x \in V$
  ⟨*proof*⟩

lemma *add-left-commute*:
  $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z)$
⟨*proof*⟩

lemmas *add-ac* = *add-assoc add-commute add-left-commute*

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

lemma *zero* [*iff*]: $0 \in V$
⟨*proof*⟩

lemma *add-zero-right* [*simp*]: $x \in V \Longrightarrow x + 0 = x$
⟨*proof*⟩

lemma *mult-assoc2*: $x \in V \Longrightarrow a \cdot b \cdot x = (a * b) \cdot x$
⟨*proof*⟩

lemma *diff-mult-distrib1*: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x - y) = a \cdot x - a \cdot y$
⟨*proof*⟩

lemma *diff-mult-distrib2*: $x \in V \Longrightarrow (a - b) \cdot x = a \cdot x - (b \cdot x)$
⟨*proof*⟩

lemmas *distrib* =
add-mult-distrib1 add-mult-distrib2
diff-mult-distrib1 diff-mult-distrib2

Further derived laws:

lemma *mult-zero-left* [*simp*]: $x \in V \Longrightarrow 0 \cdot x = 0$
⟨*proof*⟩

lemma *mult-zero-right* [*simp*]: $a \cdot 0 = (0{::}'a)$
⟨*proof*⟩

lemma *minus-mult-cancel* [*simp*]: $x \in V \Longrightarrow (- a) \cdot - x = a \cdot x$
⟨*proof*⟩

lemma *add-minus-left-eq-diff*: $x \in V \Longrightarrow y \in V \Longrightarrow - x + y = y - x$
⟨*proof*⟩

lemma *add-minus* [*simp*]: $x \in V \Longrightarrow x + - x = 0$
⟨*proof*⟩

lemma *add-minus-left* [*simp*]: $x \in V \Longrightarrow - x + x = 0$
⟨*proof*⟩

lemma *minus-minus* [*simp*]: $x \in V \Longrightarrow - (- x) = x$
⟨*proof*⟩

lemma *minus-zero* [*simp*]: $- (0{::}'a) = 0$
⟨*proof*⟩

lemma *minus-zero-iff* [*simp*]:
assumes x: $x \in V$
shows $(- x = 0) = (x = 0)$
⟨*proof*⟩

lemma *add-minus-cancel* [*simp*]: $x \in V \Longrightarrow y \in V \Longrightarrow x + (- x + y) = y$
⟨*proof*⟩

lemma *minus-add-cancel* [*simp*]: $x \in V \Longrightarrow y \in V \Longrightarrow - x + (x + y) = y$
⟨*proof*⟩

lemma *minus-add-distrib* [*simp*]: $x \in V \Longrightarrow y \in V \Longrightarrow - (x + y) = - x + - y$
⟨*proof*⟩

lemma *diff-zero* [*simp*]: $x \in V \Longrightarrow x - 0 = x$

⟨*proof*⟩

lemma *diff-zero-right* [*simp*]: $x \in V \Longrightarrow 0 - x = - x$
⟨*proof*⟩

lemma *add-left-cancel*:
assumes *x*: $x \in V$ **and** *y*: $y \in V$ **and** *z*: $z \in V$
shows $(x + y = x + z) = (y = z)$
⟨*proof*⟩

lemma *add-right-cancel*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (y + x = z + x) = (y = z)$
⟨*proof*⟩

lemma *add-assoc-cong*:
$x \in V \Longrightarrow y \in V \Longrightarrow x' \in V \Longrightarrow y' \in V \Longrightarrow z \in V$
$\Longrightarrow x + y = x' + y' \Longrightarrow x + (y + z) = x' + (y' + z)$
⟨*proof*⟩

lemma *mult-left-commute*: $x \in V \Longrightarrow a \cdot b \cdot x = b \cdot a \cdot x$
⟨*proof*⟩

lemma *mult-zero-uniq*:
assumes *x*: $x \in V$ $x \neq 0$ **and** *ax*: $a \cdot x = 0$
shows $a = 0$
⟨*proof*⟩

lemma *mult-left-cancel*:
assumes *x*: $x \in V$ **and** *y*: $y \in V$ **and** *a*: $a \neq 0$
shows $(a \cdot x = a \cdot y) = (x = y)$
⟨*proof*⟩

lemma *mult-right-cancel*:
assumes *x*: $x \in V$ **and** *neq*: $x \neq 0$
shows $(a \cdot x = b \cdot x) = (a = b)$
⟨*proof*⟩

lemma *eq-diff-eq*:
assumes *x*: $x \in V$ **and** *y*: $y \in V$ **and** *z*: $z \in V$
shows $(x = z - y) = (x + y = z)$
⟨*proof*⟩

lemma *add-minus-eq-minus*:
assumes *x*: $x \in V$ **and** *y*: $y \in V$ **and** *xy*: $x + y = 0$
shows $x = - y$
⟨*proof*⟩

lemma *add-minus-eq*:
assumes *x*: $x \in V$ **and** *y*: $y \in V$ **and** *xy*: $x - y = 0$
shows $x = y$
⟨*proof*⟩

lemma *add-diff-swap*:
assumes *vs*: $a \in V$ $b \in V$ $c \in V$ $d \in V$

and *eq*: $a + b = c + d$
shows $a - c = d - b$
⟨*proof*⟩

lemma *vs-add-cancel-21*:
assumes *vs*: $x \in V \ \ y \in V \ \ z \in V \ \ u \in V$
shows $(x + (y + z) = y + u) = (x + z = u)$
⟨*proof*⟩

lemma *add-cancel-end*:
assumes *vs*: $x \in V \ \ y \in V \ \ z \in V$
shows $(x + (y + z) = y) = (x = - z)$
⟨*proof*⟩

end

end

4 Subspaces

theory *Subspace*
imports *Vector-Space HOL−Library.Set-Algebras*
begin

4.1 Definition

A non-empty subset U of a vector space V is a *subspace* of V, iff U is closed under addition and scalar multiplication.

locale *subspace* =
fixes U :: *'a*::{*minus*, *plus*, *zero*, *uminus*} *set* **and** V
assumes *non-empty* [*iff*, *intro*]: $U \neq \{\}$
and *subset* [*iff*]: $U \subseteq V$
and *add-closed* [*iff*]: $x \in U \Longrightarrow y \in U \Longrightarrow x + y \in U$
and *mult-closed* [*iff*]: $x \in U \Longrightarrow a \cdot x \in U$

notation (*symbols*)
subspace (**infix** ‹⊴› *50*)

declare *vectorspace.intro* [*intro?*] *subspace.intro* [*intro?*]

lemma *subspace-subset* [*elim*]: $U \trianglelefteq V \Longrightarrow U \subseteq V$
⟨*proof*⟩

lemma (**in** *subspace*) *subsetD* [*iff*]: $x \in U \Longrightarrow x \in V$
⟨*proof*⟩

lemma *subspaceD* [*elim*]: $U \trianglelefteq V \Longrightarrow x \in U \Longrightarrow x \in V$
⟨*proof*⟩

lemma *rev-subspaceD* [*elim?*]: $x \in U \Longrightarrow U \trianglelefteq V \Longrightarrow x \in V$
⟨*proof*⟩

lemma (**in** *subspace*) *diff-closed* [*iff*]:

assumes *vectorspace V*
assumes *x*: $x \in U$ **and** *y*: $y \in U$
shows $x - y \in U$
⟨*proof*⟩

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

lemma (**in** *subspace*) *zero* [*intro*]:
assumes *vectorspace V*
shows $0 \in U$
⟨*proof*⟩

lemma (**in** *subspace*) *neg-closed* [*iff*]:
assumes *vectorspace V*
assumes *x*: $x \in U$
shows $- x \in U$
⟨*proof*⟩

Further derived laws: every subspace is a vector space.

lemma (**in** *subspace*) *vectorspace* [*iff*]:
assumes *vectorspace V*
shows *vectorspace U*
⟨*proof*⟩

The subspace relation is reflexive.

lemma (**in** *vectorspace*) *subspace-refl* [*intro*]: $V \trianglelefteq V$
⟨*proof*⟩

The subspace relation is transitive.

lemma (**in** *vectorspace*) *subspace-trans* [*trans*]:
$U \trianglelefteq V \Longrightarrow V \trianglelefteq W \Longrightarrow U \trianglelefteq W$
⟨*proof*⟩

4.2 Linear closure

The *linear closure* of a vector *x* is the set of all scalar multiples of *x*.

definition *lin* :: (*'a*::{*minus*,*plus*,*zero*}) ⇒ *'a set*
where $lin\ x = \{a \cdot x \mid a.\ True\}$

lemma *linI* [*intro*]: $y = a \cdot x \Longrightarrow y \in lin\ x$
⟨*proof*⟩

lemma *linI'* [*iff*]: $a \cdot x \in lin\ x$
⟨*proof*⟩

lemma *linE* [*elim*]:
assumes $x \in lin\ v$
obtains *a* :: *real* **where** $x = a \cdot v$
⟨*proof*⟩

Every vector is contained in its linear closure.

lemma (**in** *vectorspace*) *x-lin-x* [*iff*]: $x \in V \Longrightarrow x \in lin\ x$
⟨*proof*⟩

lemma (**in** *vectorspace*) *0-lin-x* [*iff*]: $x \in V \Longrightarrow 0 \in lin\ x$
⟨*proof*⟩

Any linear closure is a subspace.

lemma (**in** *vectorspace*) *lin-subspace* [*intro*]:
assumes x: $x \in V$
shows $lin\ x \trianglelefteq V$
⟨*proof*⟩

Any linear closure is a vector space.

lemma (**in** *vectorspace*) *lin-vectorspace* [*intro*]:
assumes $x \in V$
shows *vectorspace* $(lin\ x)$
⟨*proof*⟩

4.3 Sum of two vectorspaces

The *sum* of two vectorspaces U and V is the set of all sums of elements from U and V.

lemma *sum-def*: $U + V = \{u + v \mid u\ v.\ u \in U \wedge v \in V\}$
⟨*proof*⟩

lemma *sumE* [*elim*]:
$x \in U + V \Longrightarrow (\bigwedge u\ v.\ x = u + v \Longrightarrow u \in U \Longrightarrow v \in V \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

lemma *sumI* [*intro*]:
$u \in U \Longrightarrow v \in V \Longrightarrow x = u + v \Longrightarrow x \in U + V$
⟨*proof*⟩

lemma *sumI′* [*intro*]:
$u \in U \Longrightarrow v \in V \Longrightarrow u + v \in U + V$
⟨*proof*⟩

U is a subspace of $U + V$.

lemma *subspace-sum1* [*iff*]:
assumes *vectorspace* U *vectorspace* V
shows $U \trianglelefteq U + V$
⟨*proof*⟩

The sum of two subspaces is again a subspace.

lemma *sum-subspace* [*intro?*]:
assumes *subspace* U E *vectorspace* E *subspace* V E
shows $U + V \trianglelefteq E$
⟨*proof*⟩

The sum of two subspaces is a vectorspace.

lemma *sum-vs* [*intro?*]:
$U \trianglelefteq E \Longrightarrow V \trianglelefteq E \Longrightarrow$ *vectorspace* $E \Longrightarrow$ *vectorspace* $(U + V)$
⟨*proof*⟩

4.4 Direct sums

The sum of U and V is called *direct*, iff the zero element is the only common element of U and V. For every element x of the direct sum of U and V the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

lemma *decomp*:
  assumes *vectorspace E subspace U E subspace V E*
  assumes *direct*: $U \cap V = \{0\}$
    and *u1*: $u1 \in U$ **and** *u2*: $u2 \in U$
    and *v1*: $v1 \in V$ **and** *v2*: $v2 \in V$
    and *sum*: $u1 + v1 = u2 + v2$
  shows $u1 = u2 \wedge v1 = v2$
⟨*proof*⟩

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page **??**): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace H and the linear closure of x_0 the components $y \in H$ and a are uniquely determined.

lemma *decomp-H′*:
  assumes *vectorspace E subspace H E*
  assumes *y1*: $y1 \in H$ **and** *y2*: $y2 \in H$
    and *x′*: $x' \notin H \;\; x' \in E \;\; x' \neq 0$
    and *eq*: $y1 + a1 \cdot x' = y2 + a2 \cdot x'$
  shows $y1 = y2 \wedge a1 = a2$
⟨*proof*⟩

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace H and the linear closure of x' the components $y \in H$ and a are unique, it follows from $y \in H$ that $a = 0$.

lemma *decomp-H′-H*:
  assumes *vectorspace E subspace H E*
  assumes *t*: $t \in H$
    and *x′*: $x' \notin H \;\; x' \in E \;\; x' \neq 0$
  shows $(SOME\ (y, a).\ t = y + a \cdot x' \wedge y \in H) = (t, 0)$
⟨*proof*⟩

The components $y \in H$ and a in $y + a \cdot x'$ are unique, so the function h' defined by $h'\ (y + a \cdot x') = h\ y + a \cdot \xi$ is definite.

lemma *h′-definite*:
  fixes *H*
  assumes *h′-def*:
    $\bigwedge x.\ h'\ x =$
      $(let\ (y, a) = SOME\ (y, a).\ (x = y + a \cdot x' \wedge y \in H)$
       $in\ (h\ y) + a * xi)$
    and *x*: $x = y + a \cdot x'$
  assumes *vectorspace E subspace H E*
  assumes *y*: $y \in H$
    and *x′*: $x' \notin H \;\; x' \in E \;\; x' \neq 0$
  shows $h'\ x = h\ y + a * xi$
⟨*proof*⟩

end

5 Normed vector spaces

theory *Normed-Space*
imports *Subspace*
begin

5.1 Quasinorms

A *seminorm* $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogeneous and subadditive.

locale *seminorm* =
  fixes V :: $'a$::{*minus*, *plus*, *zero*, *uminus*} *set*
  fixes *norm* :: $'a \Rightarrow$ *real*   (‹‖-‖›)
  assumes *ge-zero* [*intro?*]: $x \in V \Longrightarrow 0 \leq \|x\|$
    and *abs-homogenous* [*intro?*]: $x \in V \Longrightarrow \|a \cdot x\| = |a| * \|x\|$
    and *subadditive* [*intro?*]: $x \in V \Longrightarrow y \in V \Longrightarrow \|x + y\| \leq \|x\| + \|y\|$

declare *seminorm.intro* [*intro?*]

lemma (**in** *seminorm*) *diff-subadditive*:
  assumes *vectorspace* V
  shows $x \in V \Longrightarrow y \in V \Longrightarrow \|x - y\| \leq \|x\| + \|y\|$
⟨*proof*⟩

lemma (**in** *seminorm*) *minus*:
  assumes *vectorspace* V
  shows $x \in V \Longrightarrow \|- x\| = \|x\|$
⟨*proof*⟩

5.2 Norms

A *norm* $\|\cdot\|$ is a seminorm that maps only the *0* vector to *0*.

locale *norm* = *seminorm* +
  assumes *zero-iff* [*iff*]: $x \in V \Longrightarrow (\|x\| = 0) = (x = 0)$

5.3 Normed vector spaces

A vector space together with a norm is called a *normed space*.

locale *normed-vectorspace* = *vectorspace* + *norm*

declare *normed-vectorspace.intro* [*intro?*]

lemma (**in** *normed-vectorspace*) *gt-zero* [*intro?*]:
  assumes *x*: $x \in V$ **and** *neq*: $x \neq 0$
  shows $0 < \|x\|$
⟨*proof*⟩

Any subspace of a normed vector space is again a normed vectorspace.

lemma *subspace-normed-vs* [*intro?*]:
  fixes *F E norm*
  assumes *subspace F E normed-vectorspace E norm*

shows *normed-vectorspace F norm*
⟨*proof*⟩

end

6 Linearforms

theory *Linearform*
imports *Vector-Space*
begin

A *linear form* is a function on a vector space into the reals that is additive and multiplicative.

locale *linearform* =
fixes V :: $'a$::{*minus*, *plus*, *zero*, *uminus*} *set* **and** f
assumes *add* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x + y) = f\ x + f\ y$
and *mult* [*iff*]: $x \in V \Longrightarrow f\ (a \cdot x) = a * f\ x$

declare *linearform.intro* [*intro?*]

lemma (**in** *linearform*) *neg* [*iff*]:
assumes *vectorspace V*
shows $x \in V \Longrightarrow f\ (-\ x) = -\ f\ x$
⟨*proof*⟩

lemma (**in** *linearform*) *diff* [*iff*]:
assumes *vectorspace V*
shows $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x - y) = f\ x - f\ y$
⟨*proof*⟩

Every linear form yields *0* for the *0* vector.

lemma (**in** *linearform*) *zero* [*iff*]:
assumes *vectorspace V*
shows $f\ 0 = 0$
⟨*proof*⟩

end

7 An order on functions

theory *Function-Order*
imports *Subspace Linearform*
begin

7.1 The graph of a function

We define the *graph* of a (real) function f with domain F as the set

$$\{(x, f\ x).\ x \in F\}$$

So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.

type-synonym $'a$ *graph* $= ('a \times \mathit{real})$ *set*

definition *graph* :: $'a$ *set* $\Rightarrow ('a \Rightarrow \mathit{real}) \Rightarrow 'a$ *graph*
  where *graph F f* $= \{(x, f\ x) \mid x.\ x \in F\}$

lemma *graphI* [*intro*]: $x \in F \Longrightarrow (x, f\ x) \in \mathit{graph}\ F\ f$
  ⟨*proof*⟩

lemma *graphI2* [*intro?*]: $x \in F \Longrightarrow \exists\, t \in \mathit{graph}\ F\ f.\ t = (x, f\ x)$
  ⟨*proof*⟩

lemma *graphE* [*elim?*]:
  assumes $(x, y) \in \mathit{graph}\ F\ f$
  obtains $x \in F$ **and** $y = f\ x$
  ⟨*proof*⟩

7.2 Functions ordered by domain extension

A function h' is an extension of h, iff the graph of h is a subset of the graph of h'.

lemma *graph-extI*:
  $(\bigwedge x.\ x \in H \Longrightarrow h\ x = h'\ x) \Longrightarrow H \subseteq H'$
    $\Longrightarrow \mathit{graph}\ H\ h \subseteq \mathit{graph}\ H'\ h'$
  ⟨*proof*⟩

lemma *graph-extD1* [*dest?*]: $\mathit{graph}\ H\ h \subseteq \mathit{graph}\ H'\ h' \Longrightarrow x \in H \Longrightarrow h\ x = h'\ x$
  ⟨*proof*⟩

lemma *graph-extD2* [*dest?*]: $\mathit{graph}\ H\ h \subseteq \mathit{graph}\ H'\ h' \Longrightarrow H \subseteq H'$
  ⟨*proof*⟩

7.3 Domain and function of a graph

The inverse functions to *graph* are *domain* and *funct*.

definition *domain* :: $'a$ *graph* $\Rightarrow 'a$ *set*
  where *domain g* $= \{x.\ \exists\, y.\ (x, y) \in g\}$

definition *funct* :: $'a$ *graph* $\Rightarrow ('a \Rightarrow \mathit{real})$
  where *funct g* $= (\lambda x.\ (\mathit{SOME}\ y.\ (x, y) \in g))$

The following lemma states that g is the graph of a function if the relation induced by g is unique.

lemma *graph-domain-funct*:
  assumes *uniq*: $\bigwedge x\ y\ z.\ (x, y) \in g \Longrightarrow (x, z) \in g \Longrightarrow z = y$
  shows *graph* (*domain g*) (*funct g*) $= g$
  ⟨*proof*⟩

7.4 Norm-preserving extensions of a function

Given a linear form f on the space F and a seminorm p on E. The set of all linear extensions of f, to superspaces H of F, which are bounded by p, is defined as follows.

definition
norm-pres-extensions ::
$'a$::{*plus*,*minus*,*uminus*,*zero*} *set* ⇒ ($'a$ ⇒ *real*) ⇒ $'a$ *set* ⇒ ($'a$ ⇒ *real*)
⇒ $'a$ *graph set*
where
norm-pres-extensions E p F f
= {*g*. ∃ *H h*. *g* = *graph H h*
∧ *linearform H h*
∧ $H \trianglelefteq E$
∧ $F \trianglelefteq H$
∧ *graph F f* ⊆ *graph H h*
∧ (∀ *x* ∈ *H*. *h x* ≤ *p x*)}

lemma *norm-pres-extensionE* [*elim*]:
assumes *g* ∈ *norm-pres-extensions E p F f*
obtains *H h*
where *g* = *graph H h*
and *linearform H h*
and $H \trianglelefteq E$
and $F \trianglelefteq H$
and *graph F f* ⊆ *graph H h*
and ∀ *x* ∈ *H*. *h x* ≤ *p x*
⟨*proof*⟩

lemma *norm-pres-extensionI2* [*intro*]:
linearform H h ⟹ $H \trianglelefteq E$ ⟹ $F \trianglelefteq H$
⟹ *graph F f* ⊆ *graph H h* ⟹ ∀ *x* ∈ *H*. *h x* ≤ *p x*
⟹ *graph H h* ∈ *norm-pres-extensions E p F f*
⟨*proof*⟩

lemma *norm-pres-extensionI*:
∃ *H h*. *g* = *graph H h*
∧ *linearform H h*
∧ $H \trianglelefteq E$
∧ $F \trianglelefteq H$
∧ *graph F f* ⊆ *graph H h*
∧ (∀ *x* ∈ *H*. *h x* ≤ *p x*) ⟹ *g* ∈ *norm-pres-extensions E p F f*
⟨*proof*⟩

end

8 The norm of a function

theory *Function-Norm*
imports *Normed-Space Function-Order*
begin

8.1 Continuous linear forms

A linear form f on a normed vector space $(V, \|\cdot\|)$ is *continuous*, iff it is bounded, i.e.

$$\exists\, c \in R.\ \forall\, x \in V.\ |f\, x| \le c \cdot \|x\|$$

In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

locale *continuous* = *linearform* +
  fixes *norm* :: - ⇒ *real* (‹‖-‖›)
  assumes *bounded*: $\exists c.\ \forall x \in V.\ |f\ x| \leq c * \|x\|$

declare *continuous.intro* [*intro?*] *continuous-axioms.intro* [*intro?*]

lemma *continuousI* [*intro*]:
  fixes *norm* :: - ⇒ *real* (‹‖-‖›)
  assumes *linearform V f*
  assumes *r*: $\bigwedge x.\ x \in V \Longrightarrow |f\ x| \leq c * \|x\|$
  shows *continuous V f norm*
⟨*proof*⟩

8.2 The norm of a linear form

The least real number c for which holds

$$\forall x \in V.\ |f\ x| \leq c \cdot \|x\|$$

is called the *norm* of *f*.
For non-trivial vector spaces $V \neq \{0\}$ the norm can be defined as

$$\|f\| = \sup x \neq 0.\ |f\ x|\ /\ \|x\|$$

For the case $V = \{0\}$ the supremum would be taken from an empty set. Since $\mathbb{R}$ is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be $\{\} \geq 0$ so that *fn-norm* has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be *0*, as all other elements are $\{\} \geq 0$.
Thus we define the set B where the supremum is taken from as follows:

$$\{0\} \cup \{|f\ x|\ /\ \|x\|.\ x \neq 0 \land x \in F\}$$

fn-norm is equal to the supremum of B, if the supremum exists (otherwise it is undefined).

locale *fn-norm* =
  fixes *norm* :: - ⇒ *real* (‹‖-‖›)
  fixes *B* **defines** $B\ V\ f \equiv \{0\} \cup \{|f\ x|\ /\ \|x\|\ |\ x.\ x \neq 0 \land x \in V\}$
  fixes *fn-norm* (‹‖-‖--› [*0*, *1000*] *999*)
  defines $\|f\|\text{-}V \equiv \bigsqcup (B\ V\ f)$

locale *normed-vectorspace-with-fn-norm* = *normed-vectorspace* + *fn-norm*

lemma (**in** *fn-norm*) *B-not-empty* [*intro*]: $0 \in B\ V\ f$
  ⟨*proof*⟩

The following lemma states that every continuous linear form on a normed space $(V, \|\cdot\|)$ has a function norm.

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-works*:

assumes *continuous V f norm*
shows *lub* (*B V f*) ($\|f\|$-*V*)
⟨*proof*⟩

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-ub* [*intro?*]:
assumes *continuous V f norm*
assumes *b*: $b \in B\ V\ f$
shows $b \leq \|f\|$-*V*
⟨*proof*⟩

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-leastB*:
assumes *continuous V f norm*
assumes *b*: $\bigwedge b.\ b \in B\ V\ f \Longrightarrow b \leq y$
shows $\|f\|$-$V \leq y$
⟨*proof*⟩

The norm of a continuous function is always $\geq$ *0*.

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-ge-zero* [*iff*]:
assumes *continuous V f norm*
shows $0 \leq \|f\|$-*V*
⟨*proof*⟩

The fundamental property of function norms is:

$$|f\ x| \leq \|f\| \cdot \|x\|$$

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-le-cong*:
assumes *continuous V f norm linearform V f*
assumes *x*: $x \in V$
shows $|f\ x| \leq \|f\|$-$V * \|x\|$
⟨*proof*⟩

The function norm is the least positive real number for which the following inequality holds:

$$|f\ x| \leq c \cdot \|x\|$$

lemma (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-least* [*intro?*]:
assumes *continuous V f norm*
assumes *ineq*: $\bigwedge x.\ x \in V \Longrightarrow |f\ x| \leq c * \|x\|$ **and** *ge*: $0 \leq c$
shows $\|f\|$-$V \leq c$
⟨*proof*⟩

end

9 Zorn's Lemma

theory *Zorn-Lemma*
imports *Main*
begin

Zorn's Lemmas states: if every linear ordered subset of an ordered set S has an upper bound in S, then there exists a maximal element in S. In our application,

S is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn's lemma can be modified: if S is non-empty, it suffices to show that for every non-empty chain c in S the union of c also lies in S.

theorem *Zorn′s-Lemma*:
  assumes r: $\bigwedge c.\ c \in \mathit{chains}\ S \Longrightarrow \exists\, x.\ x \in c \Longrightarrow \bigcup c \in S$
    and aS: $a \in S$
  shows $\exists\, y \in S.\ \forall\, z \in S.\ y \subseteq z \longrightarrow z = y$
$\langle \mathit{proof} \rangle$

end

Part II

Lemmas for the Proof

10 The supremum wrt. the function order

theory *Hahn-Banach-Sup-Lemmas*
imports *Function-Norm Zorn-Lemma*
begin

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let E be a real vector space with a seminorm p on E. F is a subspace of E and f a linear form on F. We consider a chain c of norm-preserving extensions of f, such that $\bigcup c = graph\ H\ h$. We will show some properties about the limit function h, i.e. the supremum of the chain c.

Let c be a chain of norm-preserving extensions of the function f and let *graph* H h be the supremum of c. Every element in H is member of one of the elements of the chain.

lemmas [*dest?*] = *chainsD*
lemmas *chainsE2* [*elim?*] = *chainsD2* [*elim-format*]

lemma *some-H′h′t*:
  assumes M: $M = norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
    and cM: $c \in chains\ M$
    and u: $graph\ H\ h = \bigcup c$
    and x: $x \in H$
  shows $\exists H'\ h'.\ graph\ H'\ h' \in c$
    $\wedge\ (x,\ h\ x) \in graph\ H'\ h'$
    $\wedge\ linearform\ H'\ h' \wedge H' \trianglelefteq E$
    $\wedge\ F \trianglelefteq H' \wedge graph\ F\ f \subseteq graph\ H'\ h'$
    $\wedge\ (\forall x \in H'.\ h'\ x \leq p\ x)$
⟨*proof*⟩

Let c be a chain of norm-preserving extensions of the function f and let *graph* H h be the supremum of c. Every element in the domain H of the supremum function is member of the domain H' of some function h', such that h extends h'.

lemma *some-H′h′*:
  assumes M: $M = norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
    and cM: $c \in chains\ M$
    and u: $graph\ H\ h = \bigcup c$
    and x: $x \in H$
  shows $\exists H'\ h'.\ x \in H' \wedge graph\ H'\ h' \subseteq graph\ H\ h$
    $\wedge\ linearform\ H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ graph\ F\ f \subseteq graph\ H'\ h' \wedge (\forall x \in H'.\ h'\ x \leq p\ x)$
⟨*proof*⟩

Any two elements x and y in the domain H of the supremum function h are both in the domain H' of some function h', such that h extends h'.

lemma *some-H′h′2*:
  assumes *M*: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    and *cM*: $c \in \textit{chains } M$
    and *u*: $\textit{graph } H\ h = \bigcup c$
    and *x*: $x \in H$
    and *y*: $y \in H$
  shows $\exists H'\ h'.\ x \in H' \wedge y \in H'$
    $\wedge\ \textit{graph } H'\ h' \subseteq \textit{graph } H\ h$
    $\wedge\ \textit{linearform } H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ \textit{graph } F\ f \subseteq \textit{graph } H'\ h' \wedge (\forall x \in H'.\ h'\ x \le p\ x)$
⟨*proof*⟩

The relation induced by the graph of the supremum of a chain c is definite, i.e. it is the graph of a function.

lemma *sup-definite*:
  assumes *M-def*: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    and *cM*: $c \in \textit{chains } M$
    and *xy*: $(x, y) \in \bigcup c$
    and *xz*: $(x, z) \in \bigcup c$
  shows $z = y$
⟨*proof*⟩

The limit function h is linear. Every element x in the domain of h is in the domain of a function h' in the chain of norm preserving extensions. Furthermore, h is an extension of h' so the function values of x are identical for h' and h. Finally, the function h' is linear by construction of M.

lemma *sup-lf*:
  assumes *M*: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    and *cM*: $c \in \textit{chains } M$
    and *u*: $\textit{graph } H\ h = \bigcup c$
  shows *linearform H h*
⟨*proof*⟩

The limit of a non-empty chain of norm preserving extensions of f is an extension of f, since every element of the chain is an extension of f and the supremum is an extension for every element of the chain.

lemma *sup-ext*:
  assumes *graph*: $\textit{graph } H\ h = \bigcup c$
    and *M*: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    and *cM*: $c \in \textit{chains } M$
    and *ex*: $\exists x.\ x \in c$
  shows $\textit{graph } F\ f \subseteq \textit{graph } H\ h$
⟨*proof*⟩

The domain H of the limit function is a superspace of F, since F is a subset of H. The existence of the *0* element in F and the closure properties follow from the fact that F is a vector space.

lemma *sup-supF*:
  assumes *graph*: $\textit{graph } H\ h = \bigcup c$
    and *M*: $M = \textit{norm-pres-extensions } E\ p\ F\ f$

and *cM*: $c \in$ *chains* M
and *ex*: $\exists x.\ x \in c$
and *FE*: $F \trianglelefteq E$
shows $F \trianglelefteq H$
⟨*proof*⟩

The domain H of the limit function is a subspace of E.

lemma *sup-subE*:
assumes *graph*: *graph* $H\ h = \bigcup c$
and *M*: $M =$ *norm-pres-extensions* $E\ p\ F\ f$
and *cM*: $c \in$ *chains* M
and *ex*: $\exists x.\ x \in c$
and *FE*: $F \trianglelefteq E$
and *E*: *vectorspace* E
shows $H \trianglelefteq E$
⟨*proof*⟩

The limit function is bounded by the norm p as well, since all elements in the chain are bounded by p.

lemma *sup-norm-pres*:
assumes *graph*: *graph* $H\ h = \bigcup c$
and *M*: $M =$ *norm-pres-extensions* $E\ p\ F\ f$
and *cM*: $c \in$ *chains* M
shows $\forall x \in H.\ h\ x \leq p\ x$
⟨*proof*⟩

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma *abs-Hahn-Banach* (see page 24). For real vector spaces the following inequality are equivalent:

$$\forall x \in H.\ |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H.\ h\ x \leq p\ x$$

lemma *abs-ineq-iff*:
assumes *subspace* $H\ E$ **and** *vectorspace* E **and** *seminorm* $E\ p$
and *linearform* $H\ h$
shows $(\forall x \in H.\ |h\ x| \leq p\ x) = (\forall x \in H.\ h\ x \leq p\ x)$ (**is** *?L* = *?R*)
⟨*proof*⟩

end

11 Extending non-maximal functions

theory *Hahn-Banach-Ext-Lemmas*
imports *Function-Norm*
begin

In this section the following context is presumed. Let E be a real vector space with a seminorm q on E. F is a subspace of E and f a linear function on F. We consider a subspace H of E that is a superspace of F and a linear form h on H. H is a not equal to E and x_0 is an element in $E - H$. H is extended to the direct sum $H' = H + lin\ x_0$, so for any $x \in H'$ the decomposition of $x = y +$

$a \cdot x$ with $y \in H$ is unique. h' is defined on H' by $h'\ x = h\ y + a \cdot \xi$ for a certain ξ.

Subsequently we show some properties of this extension h' of h.

This lemma will be used to show the existence of a linear extension of f (see page **??**). It is a consequence of the completeness of $\mathbb{R}$. To show

$$\exists \xi.\ \forall y \in F.\ a\ y \leq \xi \wedge \xi \leq b\ y$$

it suffices to show that

$$\forall u \in F.\ \forall v \in F.\ a\ u \leq b\ v$$

lemma *ex-xi*:
  assumes *vectorspace F*
  assumes *r*: $\bigwedge u\ v.\ u \in F \Longrightarrow v \in F \Longrightarrow a\ u \leq b\ v$
  shows $\exists\, xi{::}real.\ \forall y \in F.\ a\ y \leq xi \wedge xi \leq b\ y$
⟨*proof*⟩

The function h' is defined as a $h'\ x = h\ y + a \cdot \xi$ where $x = y + a \cdot \xi$ is a linear extension of h to H'.

lemma *h'-lf*:
  assumes *h'-def*: $\bigwedge x.\ h'\ x = (let\ (y, a) =$
    $SOME\ (y, a).\ x = y + a \cdot x0 \wedge y \in H\ in\ h\ y + a * xi)$
   and *H'-def*: $H' = H + lin\ x0$
   and *HE*: $H \trianglelefteq E$
  assumes *linearform H h*
  assumes *x0*: $x0 \notin H\ \ x0 \in E\ \ x0 \neq 0$
  assumes *E*: *vectorspace E*
  shows *linearform H' h'*
⟨*proof*⟩

The linear extension h' of h is bounded by the seminorm p.

lemma *h'-norm-pres*:
  assumes *h'-def*: $\bigwedge x.\ h'\ x = (let\ (y, a) =$
    $SOME\ (y, a).\ x = y + a \cdot x0 \wedge y \in H\ in\ h\ y + a * xi)$
   and *H'-def*: $H' = H + lin\ x0$
   and *x0*: $x0 \notin H\ \ x0 \in E\ \ x0 \neq 0$
  assumes *E*: *vectorspace E* **and** *HE*: *subspace H E*
   and *seminorm E p* **and** *linearform H h*
  assumes *a*: $\forall y \in H.\ h\ y \leq p\ y$
   and *a'*: $\forall y \in H.\ -\ p\ (y + x0) - h\ y \leq xi \wedge xi \leq p\ (y + x0) - h\ y$
  shows $\forall x \in H'.\ h'\ x \leq p\ x$
⟨*proof*⟩

end

Part III
The Main Proof

12 The Hahn-Banach Theorem

theory *Hahn-Banach*
imports *Hahn-Banach-Lemmas*
begin

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let F be a subspace of a real vector space E, let p be a semi-norm on E, and f be a linear form defined on F such that f is bounded by p, i.e. $\forall x \in F.\ f\ x \leq p\ x$. Then f can be extended to a linear form h on E such that h is norm-preserving, i.e. h is also bounded by p.

Proof Sketch.

1. Define M as the set of norm-preserving extensions of f to subspaces of E. The linear forms in M are ordered by domain extension.
2. We show that every non-empty chain in M has an upper bound in M.
3. With Zorn's Lemma we conclude that there is a maximal function g in M.
4. The domain H of g is the whole space E, as shown by classical contradiction:
   - Assuming g is not defined on whole E, it can still be extended in a norm-preserving way to a super-space H' of H.
   - Thus g can not be maximal. Contradiction!

theorem *Hahn-Banach*:
  assumes E: *vectorspace* E **and** *subspace* F E
    and *seminorm* E p **and** *linearform* F f
  assumes *fp*: $\forall x \in F.\ f\ x \leq p\ x$
  shows $\exists h.$ *linearform* $E\ h \wedge (\forall x \in F.\ h\ x = f\ x) \wedge (\forall x \in E.\ h\ x \leq p\ x)$
    — Let E be a vector space, F a subspace of E, p a seminorm on E,
    — and f a linear form on F such that f is bounded by p,
    — then f can be extended to a linear form h on E in a norm-preserving way.
⟨*proof*⟩

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form f and a seminorm p the following inequality are equivalent:[1]

[1] This was shown in lemma *abs-ineq-iff* (see page 22).

$$\forall x \in H.\ |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H.\ h\ x \leq p\ x$$

theorem *abs-Hahn-Banach*:
  assumes *E*: *vectorspace E* **and** *FE*: *subspace F E*
    and *lf*: *linearform F f* **and** *sn*: *seminorm E p*
  assumes *fp*: $\forall x \in F.\ |f\ x| \leq p\ x$
  shows $\exists g.$ *linearform E g*
    $\wedge (\forall x \in F.\ g\ x = f\ x)$
    $\wedge (\forall x \in E.\ |g\ x| \leq p\ x)$
⟨*proof*⟩

12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form f on a subspace F of a norm space E, can be extended to a continuous linear form g on E such that $\|f\| = \|g\|$.

theorem *norm-Hahn-Banach*:
  fixes *V* **and** *norm* (‹‖-‖›)
  fixes *B* **defines** $\bigwedge V f.\ B\ V\ f \equiv \{0\} \cup \{|f\ x|\ /\ \|x\|\ |\ x.\ x \neq 0 \wedge x \in V\}$
  fixes *fn-norm* (‹‖-‖--› [*0*, *1000*] *999*)
  defines $\bigwedge V f.\ \|f\|\text{-}V \equiv \bigsqcup(B\ V\ f)$
  assumes *E-norm*: *normed-vectorspace E norm* **and** *FE*: *subspace F E*
    and *linearform*: *linearform F f* **and** *continuous F f norm*
  shows $\exists g.$ *linearform E g*
    $\wedge$ *continuous E g norm*
    $\wedge (\forall x \in F.\ g\ x = f\ x)$
    $\wedge \|g\|\text{-}E = \|f\|\text{-}F$
⟨*proof*⟩

end

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[3] B. Nowak and A. Trybulec. Hahn-Banach theorem. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.uwb.edu.pl/JFM/Vol5/hahnban.html.