# Hoare Logic for Parallel Programs

Leonor Prensa Nieto

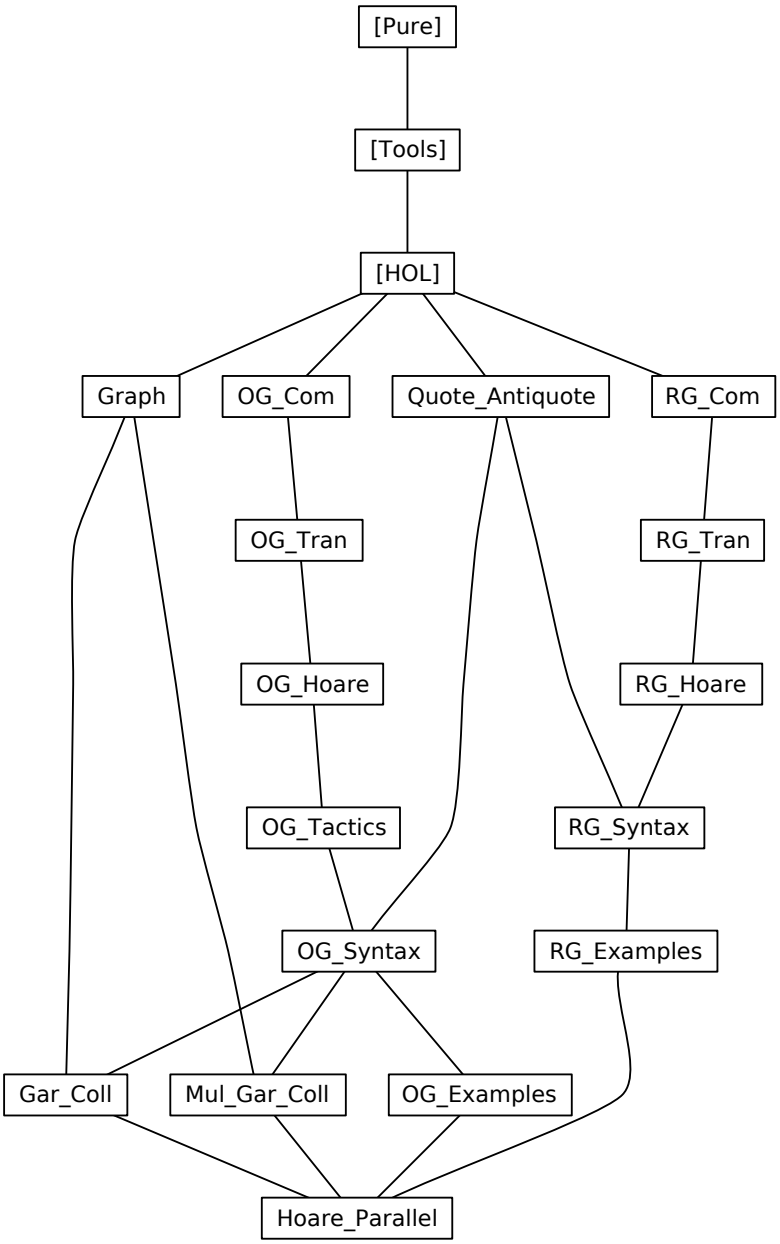
December 17, 2025

**Abstract**

In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [1].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

For excellent descriptions of this work see [2, 4, 1, 3].

# Contents


[Pure]
[Tools]
[HOL]
Graph
OG_Com
Quote_Antiquote
RG_Com
OG_Tran
RG_Tran
OG_Hoare
RG_Hoare
OG_Tactics
RG_Syntax
OG_Syntax
RG_Examples
Gar_Coll
Mul_Gar_Coll
OG_Examples
Hoare_Parallel

# Chapter 1

# The Owicki-Gries Method

## 1.1 Abstract Syntax

**theory** *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

**type-synonym** *'a bexp* = *'a set*
**type-synonym** *'a assn* = *'a set*

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

**datatype** *'a ann-com* =
    *AnnBasic* (*'a assn*) (*'a* ⇒ *'a*)
  | *AnnSeq* (*'a ann-com*) (*'a ann-com*)
  | *AnnCond1* (*'a assn*) (*'a bexp*) (*'a ann-com*) (*'a ann-com*)
  | *AnnCond2* (*'a assn*) (*'a bexp*) (*'a ann-com*)
  | *AnnWhile* (*'a assn*) (*'a bexp*) (*'a assn*) (*'a ann-com*)
  | *AnnAwait* (*'a assn*) (*'a bexp*) (*'a com*)
**and** *'a com* =
    *Parallel* (*'a ann-com option* × *'a assn*) *list*
  | *Basic* (*'a* ⇒ *'a*)
  | *Seq* (*'a com*) (*'a com*)
  | *Cond* (*'a bexp*) (*'a com*) (*'a com*)
  | *While* (*'a bexp*) (*'a assn*) (*'a com*)

The function *pre* extracts the precondition of an annotated command:

**primrec** *pre* ::*'a ann-com* ⇒ *'a assn* **where**
  *pre* (*AnnBasic r f*) = *r*
| *pre* (*AnnSeq c1 c2*) = *pre c1*
| *pre* (*AnnCond1 r b c1 c2*) = *r*
| *pre* (*AnnCond2 r b c*) = *r*
| *pre* (*AnnWhile r b i c*) = *r*
| *pre* (*AnnAwait r b c*) = *r*

Well-formedness predicate for atomic programs:

**primrec** *atom-com* :: *'a com* ⇒ *bool* **where**
  *atom-com* (*Parallel Ts*) = *False*
| *atom-com* (*Basic f*) = *True*
| *atom-com* (*Seq c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
| *atom-com* (*Cond b c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
| *atom-com* (*While b i c*) = *atom-com c*

**end**

## 1.2 Operational Semantics

**theory** *OG-Tran* **imports** *OG-Com* **begin**

**type-synonym** *'a ann-com-op* = (*'a ann-com*) *option*
**type-synonym** *'a ann-triple-op* = (*'a ann-com-op* × *'a assn*)

**primrec** *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op* **where**
  *com* (*c*, *q*) = *c*

**primrec** *post* :: *'a ann-triple-op* ⇒ *'a assn* **where**
  *post* (*c*, *q*) = *q*

**definition** *All-None* :: *'a ann-triple-op list* ⇒ *bool* **where**
  *All-None Ts* ≡ ∀(*c*, *q*) ∈ *set Ts*. *c* = *None*

### 1.2.1 The Transition Relation

**inductive-set**
  *ann-transition* :: ((*'a ann-com-op* × *'a*) × (*'a ann-com-op* × *'a*)) *set*
  **and** *transition* :: ((*'a com* × *'a*) × (*'a com* × *'a*)) *set*
  **and** *ann-transition'* :: (*'a ann-com-op* × *'a*) ⇒ (*'a ann-com-op* × *'a*) ⇒ *bool*
    (‹- −*1*→ -›[*81*,*81*] *100*)
  **and** *transition'* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
    (‹- −*P1*→ -›[*81*,*81*] *100*)
  **and** *transitions* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
    (‹- −*P*∗→ -›[*81*,*81*] *100*)
**where**
  *con-0* −*1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *ann-transition*
| *con-0* −*P1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*
| *con-0* −*P*∗→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*$^*$

| *AnnBasic*: (*Some* (*AnnBasic r f*), *s*) −*1*→ (*None*, *f s*)

| *AnnSeq1*: (*Some c0*, *s*) −*1*→ (*None*, *t*) ⟹
            (*Some* (*AnnSeq c0 c1*), *s*) −*1*→ (*Some c1*, *t*)
| *AnnSeq2*: (*Some c0*, *s*) −*1*→ (*Some c2*, *t*) ⟹
            (*Some* (*AnnSeq c0 c1*), *s*) −*1*→ (*Some* (*AnnSeq c2 c1*), *t*)

| *AnnCond1T*: *s* ∈ *b* ⟹ (*Some* (*AnnCond1 r b c1 c2*), *s*) −*1*→ (*Some c1*, *s*)

| *AnnCond1F*: $s \notin b \Longrightarrow (Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\rightarrow (Some\ c2,\ s)$

| *AnnCond2T*: $s \in b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (Some\ c,\ s)$
| *AnnCond2F*: $s \notin b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (None,\ s)$

| *AnnWhileF*: $s \notin b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s) -1\rightarrow (None,\ s)$
| *AnnWhileT*: $s \in b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s) -1\rightarrow$
$(Some\ (AnnSeq\ c\ (AnnWhile\ i\ b\ i\ c)),\ s)$

| *AnnAwait*: $[\![\ s \in b;\ atom\text{-}com\ c;\ (c,\ s) -P*\rightarrow (Parallel\ [],\ t)\ ]\!] \Longrightarrow$
$(Some\ (AnnAwait\ r\ b\ c),\ s) -1\rightarrow (None,\ t)$

| *Parallel*: $[\![\ i<length\ Ts;\ Ts!i = (Some\ c,\ q);\ (Some\ c,\ s) -1\rightarrow (r,\ t)\ ]\!]$
$\Longrightarrow (Parallel\ Ts,\ s) -P1\rightarrow (Parallel\ (Ts\ [i:=(r,\ q)]),\ t)$

| *Basic*: $(Basic\ f,\ s) -P1\rightarrow (Parallel\ [],\ f\ s)$

| *Seq1*: $All\text{-}None\ Ts \Longrightarrow (Seq\ (Parallel\ Ts)\ c,\ s) -P1\rightarrow (c,\ s)$
| *Seq2*: $(c0,\ s) -P1\rightarrow (c2,\ t) \Longrightarrow (Seq\ c0\ c1,\ s) -P1\rightarrow (Seq\ c2\ c1,\ t)$

| *CondT*: $s \in b \Longrightarrow (Cond\ b\ c1\ c2,\ s) -P1\rightarrow (c1,\ s)$
| *CondF*: $s \notin b \Longrightarrow (Cond\ b\ c1\ c2,\ s) -P1\rightarrow (c2,\ s)$

| *WhileF*: $s \notin b \Longrightarrow (While\ b\ i\ c,\ s) -P1\rightarrow (Parallel\ [],\ s)$
| *WhileT*: $s \in b \Longrightarrow (While\ b\ i\ c,\ s) -P1\rightarrow (Seq\ c\ (While\ b\ i\ c),\ s)$

**monos** *rtrancl-mono*

The corresponding abbreviations are:

**abbreviation**
*ann-transition-n* :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow nat \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a)$
$\Rightarrow bool\ (\langle\text{- } -\text{-}\rightarrow \text{-}\rangle[81,81]\ 100)$ **where**
*con-0* $-n\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition\ \widehat{\ }\ n$

**abbreviation**
*ann-transitions* :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow bool$
$(\langle\text{- } -*\rightarrow \text{-}\rangle[81,81]\ 100)$ **where**
*con-0* $-*\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition^*$

**abbreviation**
*transition-n* :: $('a\ com \times 'a) \Rightarrow nat \Rightarrow ('a\ com \times 'a) \Rightarrow bool$
$(\langle\text{- } -P\text{-}\rightarrow \text{-}\rangle[81,81,81]\ 100)$ **where**
*con-0* $-Pn\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in transition\ \widehat{\ }\ n$

### 1.2.2 Definition of Semantics

**definition** *ann-sem* :: $'a\ ann\text{-}com \Rightarrow 'a \Rightarrow 'a\ set$ **where**
*ann-sem* $c \equiv \lambda s.\ \{t.\ (Some\ c,\ s) -*\rightarrow (None,\ t)\}$

**definition** *ann-SEM* :: *'a ann-com* ⇒ *'a set* ⇒ *'a set* **where**
*ann-SEM c S* ≡ ⋃(*ann-sem c* ‘ *S*)

**definition** *sem* :: *'a com* ⇒ *'a* ⇒ *'a set* **where**
*sem c* ≡ λ*s*. {*t*. ∃ *Ts*. (*c*, *s*) −*P*∗→ (*Parallel Ts*, *t*) ∧ *All-None Ts*}

**definition** *SEM* :: *'a com* ⇒ *'a set* ⇒ *'a set* **where**
*SEM c S* ≡ ⋃(*sem c* ‘ *S*)

**abbreviation** *Omega* :: *'a com* (‹Ω› *63*)
**where** Ω ≡ *While UNIV UNIV* (*Basic id*)

**primrec** *fwhile* :: *'a bexp* ⇒ *'a com* ⇒ *nat* ⇒ *'a com* **where**
*fwhile b c 0* = Ω
| *fwhile b c* (*Suc n*) = *Cond b* (*Seq c* (*fwhile b c n*)) (*Basic id*)

## Proofs

**declare** *ann-transition-transition.intros* [*intro*]
**inductive-cases** *transition-cases*:
(*Parallel T*,*s*) −*P1*→ *t*
(*Basic f*, *s*) −*P1*→ *t*
(*Seq c1 c2*, *s*) −*P1*→ *t*
(*Cond b c1 c2*, *s*) −*P1*→ *t*
(*While b i c*, *s*) −*P1*→ *t*

**lemma** *Parallel-empty-lemma* [*rule-format* (*no-asm*)]:
(*Parallel* []*,s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=[] ∧ *n*=*0* ∧ *s*=*t*
⟨*proof*⟩

**lemma** *Parallel-AllNone-lemma* [*rule-format* (*no-asm*)]:
*All-None Ss* ⟶ (*Parallel Ss*,*s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=*Ss* ∧ *n*=*0* ∧ *s*=*t*
⟨*proof*⟩

**lemma** *Parallel-AllNone*: *All-None Ts* ⟹ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

**lemma** *Parallel-empty*: *Ts*=[] ⟹ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

**lemma** *L3-5i*: *X*⊆*Y* ⟹ *SEM c X* ⊆ *SEM c Y*
⟨*proof*⟩

**lemma** *L3-5ii-lemma1*:
⟦ (*c1*, *s1*) −*P*∗→ (*Parallel Ts*, *s2*); *All-None Ts*;
(*c2*, *s2*) −*P*∗→ (*Parallel Ss*, *s3*); *All-None Ss* ⟧
⟹ (*Seq c1 c2*, *s1*) −*P*∗→ (*Parallel Ss*, *s3*)

⟨*proof*⟩

**lemma** *L3-5ii-lemma2* [*rule-format* (*no-asm*)]:
$\forall$ *c1 c2 s t*. (*Seq c1 c2*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$
(*All-None Ts*) $\longrightarrow$ ($\exists$ *y m Rs*. (*c1*,*s*) $-P*\rightarrow$ (*Parallel Rs*, *y*) $\wedge$
(*All-None Rs*) $\wedge$ (*c2*, *y*) $-Pm\rightarrow$ (*Parallel Ts*, *t*) $\wedge$ $m \leq n$)
⟨*proof*⟩

**lemma** *L3-5ii-lemma3*:
$[\![$(*Seq c1 c2*,*s*) $-P*\rightarrow$ (*Parallel Ts*,*t*); *All-None Ts*$]\!] \Longrightarrow$
($\exists$ *y Rs*. (*c1*,*s*) $-P*\rightarrow$ (*Parallel Rs*,*y*) $\wedge$ *All-None Rs*
$\wedge$ (*c2*,*y*) $-P*\rightarrow$ (*Parallel Ts*,*t*))
⟨*proof*⟩

**lemma** *L3-5ii*: *SEM* (*Seq c1 c2*) *X* = *SEM c2* (*SEM c1 X*)
⟨*proof*⟩

**lemma** *L3-5iii*: *SEM* (*Seq* (*Seq c1 c2*) *c3*) *X* = *SEM* (*Seq c1* (*Seq c2 c3*)) *X*
⟨*proof*⟩

**lemma** *L3-5iv*:
*SEM* (*Cond b c1 c2*) *X* = (*SEM c1* (*X* $\cap$ *b*)) *Un* (*SEM c2* (*X* $\cap$ ($-b$)))
⟨*proof*⟩

**lemma** *L3-5v-lemma1*[*rule-format*]:
(*S*,*s*) $-Pn\rightarrow$ (*T*,*t*) $\longrightarrow$ *S*=$\Omega$ $\longrightarrow$ ($\neg$($\exists$ *Rs*. *T*=(*Parallel Rs*) $\wedge$ *All-None Rs*))
⟨*proof*⟩

**lemma** *L3-5v-lemma2*: $[\![$($\Omega$, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*); *All-None Ts* $]\!] \Longrightarrow$ *False*
⟨*proof*⟩

**lemma** *L3-5v-lemma3*: *SEM* ($\Omega$) *S* = {}
⟨*proof*⟩

**lemma** *L3-5v-lemma4* [*rule-format*]:
$\forall$ *s*. (*While b i c*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
($\exists$ *k*. (*fwhile b c k*, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*))
⟨*proof*⟩

**lemma** *L3-5v-lemma5* [*rule-format*]:
$\forall$ *s*. (*fwhile b c k*, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
(*While b i c*, *s*) $-P*\rightarrow$ (*Parallel Ts*,*t*)
⟨*proof*⟩

**lemma** *L3-5v*: *SEM* (*While b i c*) = ($\lambda$*x*. ($\bigcup$ *k*. *SEM* (*fwhile b c k*) *x*))
⟨*proof*⟩

## 1.3 Validity of Correctness Formulas

**definition** *com-validity* :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* (‹(*3*‖= -// -//-)› [*90*,*55*,*90*] *50*) **where**
  ‖= *p c q* ≡ *SEM c p* ⊆ *q*

**definition** *ann-com-validity* :: *'a ann-com* ⇒ *'a assn* ⇒ *bool* (‹⊨ - -› [*60*,*90*] *45*) **where**
  ⊨ *c q* ≡ *ann-SEM c* (*pre c*) ⊆ *q*

**end**

## 1.4 The Proof System

**theory** *OG-Hoare* **imports** *OG-Tran* **begin**

**primrec** *assertions* :: *'a ann-com* ⇒ (*'a assn*) *set* **where**
  *assertions* (*AnnBasic r f*) = {*r*}
| *assertions* (*AnnSeq c1 c2*) = *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond1 r b c1 c2*) = {*r*} ∪ *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond2 r b c*) = {*r*} ∪ *assertions c*
| *assertions* (*AnnWhile r b i c*) = {*r*, *i*} ∪ *assertions c*
| *assertions* (*AnnAwait r b c*) = {*r*}

**primrec** *atomics* :: *'a ann-com* ⇒ (*'a assn* × *'a com*) *set* **where**
  *atomics* (*AnnBasic r f*) = {(*r*, *Basic f*)}
| *atomics* (*AnnSeq c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond1 r b c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond2 r b c*) = *atomics c*
| *atomics* (*AnnWhile r b i c*) = *atomics c*
| *atomics* (*AnnAwait r b c*) = {(*r* ∩ *b*, *c*)}

**primrec** *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op* **where**
  *com* (*c*, *q*) = *c*

**primrec** *post* :: *'a ann-triple-op* ⇒ *'a assn* **where**
  *post* (*c*, *q*) = *q*

**definition** *interfree-aux* :: (*'a ann-com-op* × *'a assn* × *'a ann-com-op*) ⇒ *bool* **where**
  *interfree-aux* ≡ λ(*co*, *q*, *co'*). *co'*= *None* ∨
    (∀(*r*,*a*) ∈ *atomics* (*the co'*). ‖= (*q* ∩ *r*) *a q* ∧
    (*co* = *None* ∨ (∀ *p* ∈ *assertions* (*the co*). ‖= (*p* ∩ *r*) *a p*)))

**definition** *interfree* :: ((*'a ann-triple-op*) *list*) ⇒ *bool* **where**
  *interfree Ts* ≡ ∀ *i j*. *i* < *length Ts* ∧ *j* < *length Ts* ∧ *i* ≠ *j* ⟶
    *interfree-aux* (*com* (*Ts*!*i*), *post* (*Ts*!*i*), *com* (*Ts*!*j*))

**inductive**

*oghoare* :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* (‹(*3*⊨_ -//-//-)› [*90*,*55*,*90*] *50*)
**and** *ann-hoare* :: *'a ann-com* ⇒ *'a assn* ⇒ *bool* (‹(*2*⊢ -// -)› [*60*,*90*] *45*)
**where**
*AnnBasic*: $r \subseteq \{s.\ f\ s \in q\} \Longrightarrow \vdash (AnnBasic\ r\ f)\ q$

| *AnnSeq*: $[\![ \vdash c0\ pre\ c1; \vdash c1\ q ]\!] \Longrightarrow \vdash (AnnSeq\ c0\ c1)\ q$

| *AnnCond1*: $[\![ r \cap b \subseteq pre\ c1; \vdash c1\ q; r \cap -b \subseteq pre\ c2; \vdash c2\ q ]\!]$
$\Longrightarrow \vdash (AnnCond1\ r\ b\ c1\ c2)\ q$
| *AnnCond2*: $[\![ r \cap b \subseteq pre\ c; \vdash c\ q; r \cap -b \subseteq q ]\!] \Longrightarrow \vdash (AnnCond2\ r\ b\ c)\ q$

| *AnnWhile*: $[\![ r \subseteq i; i \cap b \subseteq pre\ c; \vdash c\ i; i \cap -b \subseteq q ]\!]$
$\Longrightarrow \vdash (AnnWhile\ r\ b\ i\ c)\ q$

| *AnnAwait*: $[\![ atom\text{-}com\ c; \models (r \cap b)\ c\ q ]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ c)\ q$

| *AnnConseq*: $[\![ \vdash c\ q; q \subseteq q' ]\!] \Longrightarrow \vdash c\ q'$

| *Parallel*: $[\![ \forall i{<}length\ Ts.\ \exists c\ q.\ Ts!i = (Some\ c,\ q) \wedge \vdash c\ q; interfree\ Ts ]\!]$
$\Longrightarrow \models (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts!i))))$
$Parallel\ Ts$
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts!i))$

| *Basic*: $\models \{s.\ f\ s \in q\}\ (Basic\ f)\ q$

| *Seq*: $[\![ \models p\ c1\ r; \models r\ c2\ q ]\!] \Longrightarrow \models p\ (Seq\ c1\ c2)\ q$

| *Cond*: $[\![ \models (p \cap b)\ c1\ q; \models (p \cap -b)\ c2\ q ]\!] \Longrightarrow \models p\ (Cond\ b\ c1\ c2)\ q$

| *While*: $[\![ \models (p \cap b)\ c\ p ]\!] \Longrightarrow \models p\ (While\ b\ i\ c)\ (p \cap -b)$

| *Conseq*: $[\![ p' \subseteq p; \models p\ c\ q\ ;\ q \subseteq q' ]\!] \Longrightarrow \models p'\ c\ q'$

## 1.5 Soundness

**lemmas** [*cong del*] = *if-weak-cong*

**lemmas** *ann-hoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct2*]
**lemmas** *oghoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct1*]

**lemmas** *AnnBasic* = *oghoare-ann-hoare.AnnBasic*
**lemmas** *AnnSeq* = *oghoare-ann-hoare.AnnSeq*
**lemmas** *AnnCond1* = *oghoare-ann-hoare.AnnCond1*
**lemmas** *AnnCond2* = *oghoare-ann-hoare.AnnCond2*
**lemmas** *AnnWhile* = *oghoare-ann-hoare.AnnWhile*
**lemmas** *AnnAwait* = *oghoare-ann-hoare.AnnAwait*
**lemmas** *AnnConseq* = *oghoare-ann-hoare.AnnConseq*

**lemmas** *Parallel* = *oghoare-ann-hoare.Parallel*
**lemmas** *Basic* = *oghoare-ann-hoare.Basic*
**lemmas** *Seq* = *oghoare-ann-hoare.Seq*
**lemmas** *Cond* = *oghoare-ann-hoare.Cond*
**lemmas** *While* = *oghoare-ann-hoare.While*
**lemmas** *Conseq* = *oghoare-ann-hoare.Conseq*

### 1.5.1 Soundness of the System for Atomic Programs

**lemma** *Basic-ntran* [*rule-format*]:
$(Basic\ f,\ s) -Pn\rightarrow (Parallel\ Ts,\ t) \longrightarrow All\text{-}None\ Ts \longrightarrow t = f\ s$
⟨*proof*⟩

**lemma** *SEM-fwhile*: $SEM\ S\ (p \cap b) \subseteq p \Longrightarrow SEM\ (fwhile\ b\ S\ k)\ p \subseteq (p \cap -b)$
⟨*proof*⟩

**lemma** *atom-hoare-sound* [*rule-format*]:
$\|-\ p\ c\ q \longrightarrow atom\text{-}com(c) \longrightarrow \|=\ p\ c\ q$
⟨*proof*⟩

### 1.5.2 Soundness of the System for Component Programs

**inductive-cases** *ann-transition-cases*:
$(None,s) -1\rightarrow (c',\ s')$
$(Some\ (AnnBasic\ r\ f),s) -1\rightarrow (c',\ s')$
$(Some\ (AnnSeq\ c1\ c2),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnWhile\ r\ b\ I\ c),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnAwait\ r\ b\ c),s) -1\rightarrow (c',\ s')$

Strong Soundness for Component Programs:

**lemma** *ann-hoare-case-analysis* [*rule-format*]: $\vdash\ C\ q' \longrightarrow$
$((\forall\, r\ f.\ C = AnnBasic\ r\ f \longrightarrow (\exists\, q.\ r \subseteq \{s.\ f\ s \in q\} \wedge q \subseteq q')) \wedge$
$(\forall\, c0\ c1.\ C = AnnSeq\ c0\ c1 \longrightarrow (\exists\, q.\ q \subseteq q' \wedge \vdash c0\ pre\ c1 \wedge \vdash c1\ q)) \wedge$
$(\forall\, r\ b\ c1\ c2.\ C = AnnCond1\ r\ b\ c1\ c2 \longrightarrow (\exists\, q.\ q \subseteq q' \wedge$
$r \cap b \subseteq pre\ c1 \wedge \vdash c1\ q \wedge r \cap -b \subseteq pre\ c2 \wedge \vdash c2\ q)) \wedge$
$(\forall\, r\ b\ c.\ C = AnnCond2\ r\ b\ c \longrightarrow$
$(\exists\, q.\ q \subseteq q' \wedge r \cap b \subseteq pre\ c\ \wedge \vdash c\ q \wedge r \cap -b \subseteq q)) \wedge$
$(\forall\, r\ i\ b\ c.\ C = AnnWhile\ r\ b\ i\ c \longrightarrow$
$(\exists\, q.\ q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre\ c \wedge \vdash c\ i \wedge i \cap -b \subseteq q)) \wedge$
$(\forall\, r\ b\ c.\ C = AnnAwait\ r\ b\ c \longrightarrow (\exists\, q.\ q \subseteq q' \wedge \|-\ (r \cap b)\ c\ q)))$
⟨*proof*⟩

**lemma** *Help*: $(transition \cap \{(x,y).\ True\}) = (transition)$
⟨*proof*⟩

**lemma** *Strong-Soundness-aux-aux* [*rule-format*]:
$(co,\ s) -1\rightarrow (co',\ t) \longrightarrow (\forall\, c.\ co = Some\ c \longrightarrow s \in pre\ c \longrightarrow$

$(\forall\, q.\ \vdash\ c\ q \longrightarrow (\mathit{if}\ co' = \mathit{None}\ \mathit{then}\ t{\in}q\ \mathit{else}\ t \in \mathit{pre}(\mathit{the}\ co') \wedge \vdash\ (\mathit{the}\ co')\ q\ )))$
⟨*proof*⟩

**lemma** *Strong-Soundness-aux*: ⟦ (*Some* c, s) −∗→ (co, t); s ∈ *pre* c; ⊢ c q ⟧
⟹ *if* co = *None then* t ∈ q *else* t ∈ *pre* (*the* co) ∧ ⊢ (*the* co) q
⟨*proof*⟩

**lemma** *Strong-Soundness*: ⟦ (*Some* c, s)−∗→(co, t); s ∈ *pre* c; ⊢ c q ⟧
⟹ *if* co = *None then* t∈q *else* t ∈ *pre* (*the* co)
⟨*proof*⟩

**lemma** *ann-hoare-sound*: ⊢ c q ⟹ ⊨ c q
⟨*proof*⟩

### 1.5.3 Soundness of the System for Parallel Programs

**lemma** *Parallel-length-post-P1*: (*Parallel Ts*,s) −*P1*→ (*R*′, t) ⟹
(∃ *Rs*. *R*′ = (*Parallel Rs*) ∧ (*length Rs*) = (*length Ts*) ∧
(∀ *i*. *i*<*length Ts* ⟶ *post*(*Rs* ! *i*) = *post*(*Ts* ! *i*)))
⟨*proof*⟩

**lemma** *Parallel-length-post-PStar*: (*Parallel Ts*,s) −*P*∗→ (*R*′,t) ⟹
(∃ *Rs*. *R*′ = (*Parallel Rs*) ∧ (*length Rs*) = (*length Ts*) ∧
(∀ *i*. *i*<*length Ts* ⟶ *post*(*Ts* ! *i*) = *post*(*Rs* ! *i*)))
⟨*proof*⟩

**lemma** *assertions-lemma*: *pre* c ∈ *assertions* c
⟨*proof*⟩

**lemma** *interfree-aux1* [*rule-format*]:
(c,s) −1→ (r,t) ⟶ (*interfree-aux*(*c1*, *q1*, c) ⟶ *interfree-aux*(*c1*, *q1*, r))
⟨*proof*⟩

**lemma** *interfree-aux2* [*rule-format*]:
(c,s) −1→ (r,t) ⟶ (*interfree-aux*(c, q, a) ⟶ *interfree-aux*(r, q, a) )
⟨*proof*⟩

**lemma** *interfree-lemma*: ⟦ (*Some* c, s) −1→ (r, t);*interfree Ts* ; *i*<*length Ts*;
*Ts*!*i* = (*Some* c, q) ⟧ ⟹ *interfree* (*Ts*[*i*:= (r, q)])
⟨*proof*⟩

Strong Soundness Theorem for Parallel Programs:

**lemma** *Parallel-Strong-Soundness-Seq-aux*:
⟦*interfree Ts*; *i*<*length Ts*; *com*(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*) ⟧
⟹ *interfree* (*Ts*[*i*:=(*Some c0*, *pre c1*)])
⟨*proof*⟩

**lemma** *Parallel-Strong-Soundness-Seq* [*rule-format* (*no-asm*)]:
⟦ ∀ *i*<*length Ts*. (*if com*(*Ts*!*i*) = *None then b* ∈ *post*(*Ts*!*i*)

*else* $b \in pre(the(com(Ts!i))) \wedge \vdash the(com(Ts!i))\ post(Ts!i)$);
$com(Ts\ !\ i) = Some(AnnSeq\ c0\ c1)$; $i<length\ Ts$; *interfree* $Ts$ $\rrbracket \Longrightarrow$
$(\forall ia<length\ Ts.$ (*if* $com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia) = None$
*then* $b \in post(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia)$
*else* $b \in pre(the(com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia))) \wedge$
$\vdash the(com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia))\ post(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia)))$
$\wedge$ *interfree* $(Ts[i:=(Some\ c0,\ pre\ c1)])$
⟨*proof*⟩

**lemma** *Parallel-Strong-Soundness-aux-aux* [*rule-format*]:
$(Some\ c,\ b) -1\rightarrow (co,\ t) \longrightarrow$
$(\forall Ts.\ i<length\ Ts \longrightarrow com(Ts\ !\ i) = Some\ c \longrightarrow$
$(\forall i<length\ Ts.$ (*if* $com(Ts\ !\ i) = None$ *then* $b \in post(Ts!i)$
*else* $b \in pre(the(com(Ts!i))) \wedge \vdash the(com(Ts!i))\ post(Ts!i))) \longrightarrow$
*interfree* $Ts \longrightarrow$
$(\forall j.\ j<length\ Ts \wedge i \neq j \longrightarrow$ (*if* $com(Ts!j) = None$ *then* $t \in post(Ts!j)$
*else* $t \in pre(the(com(Ts!j))) \wedge \vdash the(com(Ts!j))\ post(Ts!j))))$
⟨*proof*⟩

**lemma** *Parallel-Strong-Soundness-aux* [*rule-format*]:
$\llbracket(Ts',s) -P*\rightarrow (Rs',t);\ \ Ts' = (Parallel\ Ts);$ *interfree* $Ts;$
$\forall i.\ i<length\ Ts \longrightarrow (\exists c\ q.\ (Ts\ !\ i) = (Some\ c,\ q) \wedge s \in (pre\ c) \wedge \vdash c\ q\ ) \rrbracket \Longrightarrow$
$\forall Rs.\ Rs' = (Parallel\ Rs) \longrightarrow (\forall j.\ j<length\ Rs \longrightarrow$
(*if* $com(Rs\ !\ j) = None$ *then* $t \in post(Ts\ !\ j)$
*else* $t \in pre(the(com(Rs\ !\ j))) \wedge \vdash the(com(Rs\ !\ j))\ post(Ts\ !\ j))) \wedge$ *interfree* $Rs$
⟨*proof*⟩

**lemma** *Parallel-Strong-Soundness*:
$\llbracket(Parallel\ Ts,\ s) -P*\rightarrow (Parallel\ Rs,\ t);$ *interfree* $Ts;\ j<length\ Rs;$
$\forall i.\ i<length\ Ts \longrightarrow (\exists c\ q.\ Ts\ !\ i = (Some\ c,\ q) \wedge s \in pre\ c \wedge \vdash c\ q) \rrbracket \Longrightarrow$
*if* $com(Rs\ !\ j) = None$ *then* $t \in post(Ts\ !\ j)$ *else* $t \in pre\ (the(com(Rs\ !\ j)))$
⟨*proof*⟩

**lemma** *oghoare-sound* [*rule-format*]: $\|-\ p\ c\ q \longrightarrow \|= p\ c\ q$
⟨*proof*⟩

**end**

## 1.6 Generation of Verification Conditions

**theory** *OG-Tactics*
**imports** *OG-Hoare*
**begin**

**lemmas** *ann-hoare-intros=AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq*
**lemmas** *oghoare-intros=Parallel Basic Seq Cond While Conseq*

**lemma** *ParallelConseqRule*:

$[\![\ p \subseteq (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))));$
$\ \|{-}\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))))$
$\quad (Parallel\ Ts)$
$\quad (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i));$
$\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i))\subseteq q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Parallel\ Ts)\ q$
⟨*proof*⟩

**lemma** *SkipRule*: $p \subseteq q \Longrightarrow \|{-}\ p\ (Basic\ id)\ q$
⟨*proof*⟩

**lemma** *BasicRule*: $p \subseteq \{s.\ (f\ s){\in}q\} \Longrightarrow \|{-}\ p\ (Basic\ f)\ q$
⟨*proof*⟩

**lemma** *SeqRule*: $[\![\ \|{-}\ p\ c1\ r;\ \|{-}\ r\ c2\ q\ ]\!] \Longrightarrow \|{-}\ p\ (Seq\ c1\ c2)\ q$
⟨*proof*⟩

**lemma** *CondRule*:
$[\![\ p \subseteq \{s.\ (s{\in}b \longrightarrow s{\in}w) \wedge (s{\notin}b \longrightarrow s{\in}w')\};\ \|{-}\ w\ c1\ q;\ \|{-}\ w'\ c2\ q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Cond\ b\ c1\ c2)\ q$
⟨*proof*⟩

**lemma** *WhileRule*: $[\![\ p \subseteq i;\ \|{-}\ (i \cap b)\ c\ i\ ;\ (i \cap (-b)) \subseteq q\ ]\!]$
$\quad \Longrightarrow \|{-}\ p\ (While\ b\ i\ c)\ q$
⟨*proof*⟩

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is $\{s.\ True\}$:

**lemma** *AnnatomRule*:
$[\![\ atom\text{-}com(c);\ \|{-}\ r\ c\ q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ \{s.\ True\}\ c)\ q$
⟨*proof*⟩

**lemma** *AnnskipRule*:
$r \subseteq q \Longrightarrow \vdash (AnnBasic\ r\ id)\ q$
⟨*proof*⟩

**lemma** *AnnwaitRule*:
$[\![\ (r \cap b) \subseteq q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ (Basic\ id))\ q$
⟨*proof*⟩

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

**lemma** *interfree-aux-rule1*: *interfree-aux*(*co*, *q*, *None*)
⟨*proof*⟩

**lemma** *interfree-aux-rule2*:
$\forall (R,r){\in}(atomics\ a).\ \|{-}\ (q \cap R)\ r\ q \Longrightarrow interfree\text{-}aux(None,\ q,\ Some\ a)$

⟨*proof*⟩

**lemma** *interfree-aux-rule3*:
(∀ (*R*, *r*)∈(*atomics a*). ‖− (*q* ∩ *R*) *r q* ∧ (∀ *p*∈(*assertions c*). ‖− (*p* ∩ *R*) *r p*))
⟹ *interfree-aux*(*Some c*, *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnBasic-assertions*:
⟦*interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧ ⟹
*interfree-aux*(*Some* (*AnnBasic r f*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnSeq-assertions*:
⟦ *interfree-aux*(*Some c1*, *q*, *Some a*); *interfree-aux*(*Some c2*, *q*, *Some a*)⟧⟹
*interfree-aux*(*Some* (*AnnSeq c1 c2*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnCond1-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c1*, *q*, *Some a*);
*interfree-aux*(*Some c2*, *q*, *Some a*)⟧⟹
*interfree-aux*(*Some*(*AnnCond1 r b c1 c2*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnCond2-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c*, *q*, *Some a*)⟧⟹
*interfree-aux*(*Some* (*AnnCond2 r b c*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnWhile-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *i*, *Some a*);
*interfree-aux*(*Some c*, *q*, *Some a*)⟧⟹
*interfree-aux*(*Some* (*AnnWhile r b i c*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnAwait-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧⟹
*interfree-aux*(*Some* (*AnnAwait r b c*), *q*, *Some a*)
⟨*proof*⟩

**lemma** *AnnBasic-atomics*:
‖− (*q* ∩ *r*) (*Basic f*) *q* ⟹ *interfree-aux*(*None*, *q*, *Some* (*AnnBasic r f*))
⟨*proof*⟩

**lemma** *AnnSeq-atomics*:
⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
*interfree-aux*(*Any*, *q*, *Some* (*AnnSeq a1 a2*))
⟨*proof*⟩

**lemma** *AnnCond1-atomics*:

⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
*interfree-aux*(*Any*, *q*, *Some* (*AnnCond1 r b a1 a2*))
⟨*proof*⟩

**lemma** *AnnCond2-atomics*:
*interfree-aux* (*Any*, *q*, *Some a*)⟹ *interfree-aux*(*Any*, *q*, *Some* (*AnnCond2 r b a*))
⟨*proof*⟩

**lemma** *AnnWhile-atomics*: *interfree-aux* (*Any*, *q*, *Some a*)
⟹ *interfree-aux*(*Any*, *q*, *Some* (*AnnWhile r b i a*))
⟨*proof*⟩

**lemma** *Annatom-atomics*:
‖− (*q* ∩ *r*) *a q* ⟹ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r* {*x*. *True*} *a*))
⟨*proof*⟩

**lemma** *AnnAwait-atomics*:
‖− (*q* ∩ (*r* ∩ *b*)) *a q* ⟹ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r b a*))
⟨*proof*⟩

**definition** *interfree-swap* :: ('*a ann-triple-op* ∗ ('*a ann-triple-op*) *list*) ⇒ *bool* **where**
*interfree-swap* == λ(*x*, *xs*). ∀ *y*∈*set xs*. *interfree-aux* (*com x*, *post x*, *com y*)
∧ *interfree-aux*(*com y*, *post y*, *com x*)

**lemma** *interfree-swap-Empty*: *interfree-swap* (*x*, [])
⟨*proof*⟩

**lemma** *interfree-swap-List*:
⟦ *interfree-aux* (*com x*, *post x*, *com y*);
*interfree-aux* (*com y*, *post y* ,*com x*); *interfree-swap* (*x*, *xs*) ⟧
⟹ *interfree-swap* (*x*, *y*#*xs*)
⟨*proof*⟩

**lemma** *interfree-swap-Map*: ∀ *k*. *i*≤*k* ∧ *k*<*j* ⟶ *interfree-aux* (*com x*, *post x*, *c k*)
∧ *interfree-aux* (*c k*, *Q k*, *com x*)
⟹ *interfree-swap* (*x*, *map* (λ*k*. (*c k*, *Q k*)) [*i*..<*j*])
⟨*proof*⟩

**lemma** *interfree-Empty*: *interfree* []
⟨*proof*⟩

**lemma** *interfree-List*:
⟦ *interfree-swap*(*x*, *xs*); *interfree xs* ⟧ ⟹ *interfree* (*x*#*xs*)
⟨*proof*⟩

**lemma** *interfree-Map*:
(∀ *i j*. *a*≤*i* ∧ *i*<*b* ∧ *a*≤*j* ∧ *j*<*b* ∧ *i*≠*j* ⟶ *interfree-aux* (*c i*, *Q i*, *c j*))
⟹ *interfree* (*map* (λ*k*. (*c k*, *Q k*)) [*a*..<*b*])

⟨*proof*⟩

**definition** *map-ann-hoare* :: (('*a ann-com-op* ∗ '*a assn*) *list*) ⇒ *bool* (‹[⊢] -› [*0*] *45*) **where**
[⊢] *Ts* == (∀ *i*<*length Ts*. ∃ *c q*. *Ts*!*i*=(*Some c*, *q*) ∧ ⊢ *c q*)

**lemma** *MapAnnEmpty*: [⊢] []
⟨*proof*⟩

**lemma** *MapAnnList*: ⟦ ⊢ *c q* ; [⊢] *xs* ⟧ ⟹ [⊢] (*Some c*,*q*)#*xs*
⟨*proof*⟩

**lemma** *MapAnnMap*:
∀ *k*. *i*≤*k* ∧ *k*<*j* ⟶ ⊢ (*c k*) (*Q k*) ⟹ [⊢] *map* (λ*k*. (*Some* (*c k*), *Q k*)) [*i*..<*j*]
⟨*proof*⟩

**lemma** *ParallelRule*:⟦ [⊢] *Ts* ; *interfree Ts* ⟧
⟹ ||− (⋂ *i*∈{*i*. *i*<*length Ts*}. *pre*(*the*(*com*(*Ts*!*i*))))
*Parallel Ts*
(⋂ *i*∈{*i*. *i*<*length Ts*}. *post*(*Ts*!*i*))
⟨*proof*⟩

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

**lemma** *Compl-Collect*: −(*Collect b*) = {*x*. ¬(*b x*)}
⟨*proof*⟩

**lemma** *list-length*: *length* []=*0 length* (*x*#*xs*) = *Suc*(*length xs*)
⟨*proof*⟩
**lemma** *list-lemmas*: *length* []=*0 length* (*x*#*xs*) = *Suc*(*length xs*)
(*x*#*xs*) ! *0* = *x* (*x*#*xs*) ! *Suc n* = *xs* ! *n*
⟨*proof*⟩
**lemma** *le-Suc-eq-insert*: {*i*. *i* <*Suc n*} = *insert n* {*i*. *i*< *n*}
⟨*proof*⟩
**lemmas** *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*
**lemmas** *my-simp-list* = *list-lemmas fst-conv snd-conv*
*not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc nat.inject*
*Collect-mem-eq ball-simps option.simps primrecdef-list*
**lemmas** *ParallelConseq-list* = *INTER-eq Collect-conj-eq length-map length-upt length-append*

⟨*ML*⟩

The following tactic applies *tac* to each conjunct in a subgoal of the form *A* ⟹ *a1* ∧ *a2* ∧ .. ∧ *an* returning *n* subgoals, one for each conjunct:

⟨*ML*⟩

**Tactic for the generation of the verification conditions**

The tactic basically uses two subtactics:

**HoareRuleTac** is called at the level of parallel programs, it uses the ParallelTac to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. $\langle\ \rangle$ and *AWAIT b THEN - END*, and at each interference freedom test.

**AnnHoareRuleTac** is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter precond, see NOTE).

NOTE: precond(::bool) informs if the subgoal has the form $\|-$ *?p c q*, in this case we have precond=False and the generated verification condition would have the form *?p* $\subseteq \ldots$ which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

⟨*ML*⟩

The final tactic is given the name *oghoare*:

⟨*ML*⟩

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

⟨*ML*⟩

The so defined ML tactics are then "exported" to be used in Isabelle proofs.

⟨*ML*⟩

Tactics useful for dealing with the generated verification conditions:

⟨*ML*⟩

**end**

## 1.7 Concrete Syntax

**theory** *Quote-Antiquote* **imports** *Main* **begin**

**syntax**

| | | |
|---|---|---|
| *-quote* | :: $'b \Rightarrow ('a \Rightarrow 'b)$ | $(‹(«\text{-}»)› [0]\ 1000)$ |
| *-antiquote* | :: $('a \Rightarrow 'b) \Rightarrow 'b$ | $(‹'\text{-}› [1000]\ 1000)$ |

*-Assert* :: *'a ⇒ 'a set* (‹(⦃-⦄)› [0] 1000)

**translations**
⦃*b*⦄ ⇀ *CONST Collect «b»*

⟨*ML*⟩

**end**
**theory** *OG-Syntax*
**imports** *OG-Tactics Quote-Antiquote*
**begin**

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

**abbreviation** *Skip* :: *'a com* (‹*SKIP*› 63)
**where** *SKIP* ≡ *Basic id*

**abbreviation** *AnnSkip* :: *'a assn ⇒ 'a ann-com* (‹-//*SKIP*› [90] 63)
**where** *r SKIP* ≡ *AnnBasic r id*

**notation**
*Seq* (‹-,,/ -› [55, 56] 55) **and**
*AnnSeq* (‹-;;/ -› [60,61] 60)

**syntax**
*-Assign* :: *idt ⇒ 'b ⇒ 'a com* (‹(´- :=/ -)› [70, 65] 61)
*-AnnAssign* :: *'a assn ⇒ idt ⇒ 'b ⇒ 'a com* (‹(- ´- :=/ -)› [90,70,65] 61)

**translations**
*´x := a* ⇀ *CONST Basic «´(-update-name x (λ-. a))»*
*r ´x := a* ⇀ *CONST AnnBasic r «´(-update-name x (λ-. a))»*

**syntax**
*-AnnCond1* :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com ⇒ 'a ann-com*
(‹- //IF - /THEN - /ELSE - /FI› [90,0,0,0] 61)
*-AnnCond2* :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com*
(‹- //IF - /THEN - /FI› [90,0,0] 61)
*-AnnWhile* :: *'a assn ⇒ 'a bexp ⇒ 'a assn ⇒ 'a ann-com ⇒ 'a ann-com*
(‹- //WHILE - /INV - //DO -//OD› [90,0,0,0] 61)
*-AnnAwait* :: *'a assn ⇒ 'a bexp ⇒ 'a com ⇒ 'a ann-com*
(‹- //AWAIT - /THEN /- /END› [90,0,0] 61)
*-AnnAtom* :: *'a assn ⇒ 'a com ⇒ 'a ann-com* (‹-//⟨-⟩› [90,0] 61)
*-AnnWait* :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com* (‹-//WAIT - END› [90,0] 61)

*-Cond* :: *'a bexp ⇒ 'a com ⇒ 'a com ⇒ 'a com*
(‹(0IF -/ THEN -/ ELSE -/ FI)› [0, 0, 0] 61)
*-Cond2* :: *'a bexp ⇒ 'a com ⇒ 'a com* (‹IF - THEN - FI› [0,0] 56)
*-While-inv* :: *'a bexp ⇒ 'a assn ⇒ 'a com ⇒ 'a com*
(‹(0WHILE -/ INV - //DO - /OD)› [0, 0, 0] 61)

*-While* :: *'a bexp ⇒ 'a com ⇒ 'a com*
(‹(*0WHILE - //DO - /OD*)› [*0*, *0*] *61*)

**translations**
*IF b THEN c1 ELSE c2 FI ⇀ CONST Cond ⦃b⦄ c1 c2*
*IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI*
*WHILE b INV i DO c OD ⇀ CONST While ⦃b⦄ i c*
*WHILE b DO c OD ⇌ WHILE b INV CONST undefined DO c OD*

*r IF b THEN c1 ELSE c2 FI ⇀ CONST AnnCond1 r ⦃b⦄ c1 c2*
*r IF b THEN c FI ⇀ CONST AnnCond2 r ⦃b⦄ c*
*r WHILE b INV i DO c OD ⇀ CONST AnnWhile r ⦃b⦄ i c*
*r AWAIT b THEN c END ⇀ CONST AnnAwait r ⦃b⦄ c*
*r ⟨c⟩ ⇌ r AWAIT CONST True THEN c END*
*r WAIT b END ⇌ r AWAIT b THEN SKIP END*

**nonterminal** *prgs*

**syntax**
*-PAR* :: *prgs ⇒ 'a* (‹*COBEGIN//-//COEND*› [*57*] *56*)
*-prg* :: [*'a*, *'a*] ⇒ *prgs* (‹*-//-*› [*60*, *90*] *57*)
*-prgs* :: [*'a*, *'a*, *prgs*] ⇒ *prgs* (‹*-//-//‖//-*› [*60*,*90*,*57*] *57*)

*-prg-scheme* :: [*'a*, *'a*, *'a*, *'a*, *'a*] ⇒ *prgs*
(‹*SCHEME* [*- ≤ - < -*] *-// -*› [*0*,*0*,*0*,*60*, *90*] *57*)

**translations**
*-prg c q ⇌ [(CONST Some c, q)]*
*-prgs c q ps ⇌ (CONST Some c, q) # ps*
*-PAR ps ⇌ CONST Parallel ps*

*-prg-scheme j i k c q ⇌ CONST map (λi. (CONST Some c, q)) [j..<k]*

⟨*ML*⟩

**end**

# 1.8 Examples

**theory** *OG-Examples* **imports** *OG-Syntax* **begin**

## 1.8.1 Mutual Exclusion

### Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

**record** *Petersons-mutex-1* =
*pr1* :: *nat*
*pr2* :: *nat*

```
 in1 :: bool
 in2 :: bool
 hold :: nat

lemma Petersons-mutex-1:
  ||- {| ´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 |}
  COBEGIN {| ´pr1=0 ∧ ¬´in1 |}
  WHILE True INV {| ´pr1=0 ∧ ¬´in1 |}
  DO
  {| ´pr1=0 ∧ ¬´in1 |} ⟨ ´in1:=True,,´pr1:=1 ⟩;;
  {| ´pr1=1 ∧ ´in1 |} ⟨ ´hold:=1,,´pr1:=2 ⟩;;
  {| ´pr1=2 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2) |}
  AWAIT (¬´in2 ∨ ¬(´hold=1)) THEN ´pr1:=3 END;;
  {| ´pr1=3 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2) |}
   ⟨´in1:=False,,´pr1:=0⟩
  OD {| ´pr1=0 ∧ ¬´in1 |}
  ||
  {| ´pr2=0 ∧ ¬´in2 |}
  WHILE True INV {| ´pr2=0 ∧ ¬´in2 |}
  DO
  {| ´pr2=0 ∧ ¬´in2 |} ⟨ ´in2:=True,,´pr2:=1 ⟩;;
  {| ´pr2=1 ∧ ´in2 |} ⟨ ´hold:=2,,´pr2:=2 ⟩;;
  {| ´pr2=2 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2)) |}
  AWAIT (¬´in1 ∨ ¬(´hold=2)) THEN ´pr2:=3 END;;
  {| ´pr2=3 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2)) |}
    ⟨´in2:=False,,´pr2:=0⟩
  OD {| ´pr2=0 ∧ ¬´in2 |}
  COEND
  {| ´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 |}
⟨proof⟩
```

### Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

```
record Busy-wait-mutex =
 flag1 :: bool
 flag2 :: bool
 turn  :: nat
 after1 :: bool
 after2 :: bool

lemma Busy-wait-mutex:
 ||-  {| True |}
  ´flag1:=False,, ´flag2:=False,,
  COBEGIN {| ¬´flag1 |}
        WHILE True
        INV {| ¬´flag1 |}
        DO {| ¬´flag1 |} ⟨ ´flag1:=True,,´after1:=False ⟩;;
```

```
          {´flag1 ∧ ¬´after1} ⟨ ´turn:=1,,´after1:=True ⟩;;
          {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}
           WHILE ¬(´flag2 ⟶ ´turn=2)
           INV {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}
           DO {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)} SKIP OD;;
          {´flag1 ∧ ´after1 ∧ (´flag2 ∧ ´after2 ⟶ ´turn=2)}
           ´flag1:=False
       OD
      {False}
  ||
    {¬´flag2}
       WHILE True
       INV {¬´flag2}
       DO {¬´flag2} ⟨ ´flag2:=True,,´after2:=False ⟩;;
          {´flag2 ∧ ¬´after2} ⟨ ´turn:=2,,´after2:=True ⟩;;
          {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}
           WHILE ¬(´flag1 ⟶ ´turn=1)
           INV {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}
           DO {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)} SKIP OD;;
          {´flag2 ∧ ´after2 ∧ (´flag1 ∧ ´after1 ⟶ ´turn=1)}
           ´flag2:=False
       OD
      {False}
  COEND
  {False}
⟨proof⟩
```

### Peterson's Algorithm III: A Solution using Semaphores

```
record Semaphores-mutex =
 out :: bool
 who :: nat

lemma Semaphores-mutex:
 ||− {i≠j}
  ´out:=True ,,
  COBEGIN {i≠j}
       WHILE True INV {i≠j}
       DO {i≠j} AWAIT ´out THEN ´out:=False,, ´who:=i END;;
          {¬´out ∧ ´who=i ∧ i≠j} ´out:=True OD
       {False}
  ||
       {i≠j}
       WHILE True INV {i≠j}
       DO {i≠j} AWAIT ´out THEN ´out:=False,,´who:=j END;;
          {¬´out ∧ ´who=j ∧ i≠j} ´out:=True OD
       {False}
  COEND
  {False}
```

⟨*proof*⟩

### Peterson's Algorithm III: Parameterized version:

**lemma** *Semaphores-parameterized-mutex*:
*0<n* ⟹ ||− {|*True*|}
´*out*:=*True* ,,
*COBEGIN*
*SCHEME* [*0*≤ *i*< *n*]
{|*True*|}
*WHILE True INV* {|*True*|}
*DO* {|*True*|} *AWAIT* ´*out THEN* ´*out*:=*False*,, ´*who*:=*i END*;;
{|¬´*out* ∧ ´*who*=*i*|} ´*out*:=*True OD*
{|*False*|}
*COEND*
{|*False*|}
⟨*proof*⟩

### The Ticket Algorithm

**record** *Ticket-mutex* =
*num* :: *nat*
*nextv* :: *nat*
*turn* :: *nat list*
*index* :: *nat*

**lemma** *Ticket-mutex*:
⟦ *0<n*; *I*=«*n*=*length* ´*turn* ∧ *0*<´*nextv* ∧ (∀ *k l*. *k*<*n* ∧ *l*<*n* ∧ *k*≠*l*
⟶ ´*turn*!*k* < ´*num* ∧ (´*turn*!*k* =*0* ∨ ´*turn*!*k*≠´*turn*!*l*))» ⟧
⟹ ||− {|*n*=*length* ´*turn*|}
´*index*:= *0*,,
*WHILE* ´*index* < *n INV* {|*n*=*length* ´*turn* ∧ (∀ *i*<´*index*. ´*turn*!*i*=*0*)|}
*DO* ´*turn*:= ´*turn*[´*index*:=*0*],, ´*index*:=´*index* +*1 OD*,,
´*num*:=*1* ,, ´*nextv*:=*1* ,,
*COBEGIN*
*SCHEME* [*0*≤ *i*< *n*]
{|´*I*|}
*WHILE True INV* {|´*I*|}
*DO* {|´*I*|} ⟨ ´*turn* :=´*turn*[*i*:=´*num*],, ´*num*:=´*num*+*1* ⟩;;
{|´*I*|} *WAIT* ´*turn*!*i*=´*nextv END*;;
{|´*I* ∧ ´*turn*!*i*=´*nextv*|} ´*nextv*:=´*nextv*+*1*
*OD*
{|*False*|}
*COEND*
{|*False*|}
⟨*proof*⟩

## 1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

Apt and Olderog. "Verification of sequential and concurrent Programs" page 294:

**record** *Zero-search* =
  *turn* :: *nat*
  *found* :: *bool*
  *x* :: *nat*
  *y* :: *nat*

**lemma** *Zero-search*:
 ⟦*I1*= « *a*≤´*x* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
    ∧ (¬´*found* ∧ *a*<´ *x* ⟶ *f*(´*x*)≠*0*) » ;
   *I2*= «´*y*≤*a*+*1* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
    ∧ (¬´*found* ∧ ´*y*≤*a* ⟶ *f*(´*y*)≠*0*) » ⟧ ⟹
 ‖− {|∃ *u*. *f*(*u*)=*0*|}
 ´*turn*:=*1*,, ´*found*:= *False*,,
 ´*x*:=*a*,, ´*y*:=*a*+*1* ,,
 *COBEGIN* {|´*I1*|}
     *WHILE* ¬´*found*
     *INV* {|´*I1*|}
     *DO* {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        *WAIT* ´*turn*=*1* *END*;;
        {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        ´*turn*:=*2*;;
        {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        ⟨ ´*x*:=´*x*+*1*,,
          *IF* *f*(´*x*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
     *OD*;;
     {|´*I1* ∧ ´*found*|}
     ´*turn*:=*2*
     {|´*I1* ∧ ´*found*|}
 ‖
    {|´*I2*|}
     *WHILE* ¬´*found*
     *INV* {|´*I2*|}
     *DO* {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        *WAIT* ´*turn*=*2* *END*;;
        {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        ´*turn*:=*1*;;
        {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        ⟨ ´*y*:=(´*y* − *1*),,
          *IF* *f*(´*y*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
     *OD*;;
     {|´*I2* ∧ ´*found*|}
     ´*turn*:=*1*
     {|´*I2* ∧ ´*found*|}
 *COEND*
 {|*f*(´*x*)=*0* ∨ *f*(´*y*)=*0*|}
⟨*proof*⟩

Easier Version: without AWAIT. Apt and Olderog. page 256:

**lemma** *Zero-Search-2*:
$[\![ I1 = \ll a \leq \acute{} x \wedge (\acute{} found \longrightarrow (a < \acute{} x \wedge f(\acute{} x) = 0) \vee (\acute{} y \leq a \wedge f(\acute{} y) = 0))$
$\wedge (\neg \acute{} found \wedge a < \acute{} x \longrightarrow f(\acute{} x) \neq 0) \gg;$
$I2 = \ll \acute{} y \leq a+1 \wedge (\acute{} found \longrightarrow (a < \acute{} x \wedge f(\acute{} x) = 0) \vee (\acute{} y \leq a \wedge f(\acute{} y) = 0))$
$\wedge (\neg \acute{} found \wedge \acute{} y \leq a \longrightarrow f(\acute{} y) \neq 0) \gg ]\!] \Longrightarrow$
$\| - \{\!| \exists u.\ f(u) = 0 |\!\}$
$\acute{} found := False,,$
$\acute{} x := a,,\ \acute{} y := a+1,,$
$COBEGIN\ \{\!| \acute{} I1 |\!\}$
$WHILE\ \neg \acute{} found$
$INV\ \{\!| \acute{} I1 |\!\}$
$DO\ \{\!| a \leq \acute{} x \wedge (\acute{} found \longrightarrow \acute{} y \leq a \wedge f(\acute{} y) = 0) \wedge (a < \acute{} x \longrightarrow f(\acute{} x) \neq 0) |\!\}$
$\langle\ \acute{} x := \acute{} x + 1,, IF\ f(\acute{} x) = 0\ THEN\ \acute{} found := True\ ELSE\ SKIP\ FI \rangle$
$OD$
$\{\!| \acute{} I1 \wedge \acute{} found |\!\}$
$\|$
$\{\!| \acute{} I2 |\!\}$
$WHILE\ \neg \acute{} found$
$INV\ \{\!| \acute{} I2 |\!\}$
$DO\ \{\!| \acute{} y \leq a+1 \wedge (\acute{} found \longrightarrow a < \acute{} x \wedge f(\acute{} x) = 0) \wedge (\acute{} y \leq a \longrightarrow f(\acute{} y) \neq 0) |\!\}$
$\langle\ \acute{} y := (\acute{} y - 1),, IF\ f(\acute{} y) = 0\ THEN\ \acute{} found := True\ ELSE\ SKIP\ FI \rangle$
$OD$
$\{\!| \acute{} I2 \wedge \acute{} found |\!\}$
$COEND$
$\{\!| f(\acute{} x) = 0 \vee f(\acute{} y) = 0 |\!\}$
$\langle proof \rangle$

### 1.8.3 Producer/Consumer

#### Previous lemmas

**lemma** *nat-lemma2*: $[\![\ b = m*(n::nat) + t;\ a = s*n + u;\ t = u;\ b - a < n\ ]\!] \Longrightarrow m \leq s$
$\langle proof \rangle$

**lemma** *mod-lemma*: $[\![\ (c::nat) \leq a;\ a < b;\ b - c < n\ ]\!] \Longrightarrow b\ mod\ n \neq a\ mod\ n$
$\langle proof \rangle$

#### Producer/Consumer Algorithm

**record** *Producer-consumer* =
*ins* :: *nat*
*outs* :: *nat*
*li* :: *nat*
*lj* :: *nat*
*vx* :: *nat*
*vy* :: *nat*
*buffer* :: *nat list*
*b* :: *nat list*

The whole proof takes aprox. 4 minutes.

**lemma** *Producer-consumer*:

```
⟦INIT= «0<length a ∧ 0<length ´buffer ∧ length ´b=length a» ;
  I= «(∀k<´ins. ´outs≤k ⟶ (a ! k) = ´buffer ! (k mod (length ´buffer))) ∧
        ´outs≤´ins ∧ ´ins−´outs≤length ´buffer» ;
  I1= «´I ∧ ´li≤length a» ;
  p1= «´I1 ∧ ´li=´ins» ;
  I2 = «´I ∧ (∀k<´lj. (a ! k)=(´b ! k)) ∧ ´lj≤length a» ;
  p2 = «´I2 ∧ ´lj=´outs» ⟧ ⟹
 ||− {|´INIT|}
´ins:=0,, ´outs:=0,, ´li:=0,, ´lj:=0,,
 COBEGIN {|´p1 ∧ ´INIT|}
  WHILE ´li <length a
   INV {|´p1 ∧ ´INIT|}
  DO {|´p1 ∧ ´INIT ∧ ´li<length a|}
     ´vx:= (a ! ´li);;
    {|´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)|}
     WAIT ´ins−´outs < length ´buffer END;;
    {|´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)
      ∧ ´ins−´outs < length ´buffer|}
     ´buffer:=(list-update ´buffer (´ins mod (length ´buffer)) ´vx);;
    {|´p1 ∧ ´INIT ∧ ´li<length a
      ∧ (a ! ´li)=(´buffer ! (´ins mod (length ´buffer)))
      ∧ ´ins−´outs <length ´buffer|}
     ´ins:=´ins+1;;
    {|´I1 ∧ ´INIT ∧ (´li+1)=´ins ∧ ´li<length a|}
     ´li:=´li+1
  OD
 {|´p1 ∧ ´INIT ∧ ´li=length a|}
 ||
 {|´p2 ∧ ´INIT|}
  WHILE ´lj < length a
   INV {|´p2 ∧ ´INIT|}
  DO {|´p2 ∧ ´lj<length a ∧ ´INIT|}
     WAIT ´outs<´ins END;;
    {|´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´INIT|}
     ´vy:=(´buffer ! (´outs mod (length ´buffer)));;
    {|´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´vy=(a ! ´lj) ∧ ´INIT|}
     ´outs:=´outs+1;;
    {|´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ ´vy=(a ! ´lj) ∧ ´INIT|}
     ´b:=(list-update ´b ´lj ´vy);;
    {|´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ (a ! ´lj)=(´b ! ´lj) ∧ ´INIT|}
     ´lj:=´lj+1
  OD
 {|´p2 ∧ ´lj=length a ∧ ´INIT|}
 COEND
 {| ∀k<length a. (a ! k)=(´b ! k)|}
⟨proof⟩
```

### 1.8.4 Parameterized Examples

#### Set Elements of an Array to Zero

**record** *Example1* =
  $a :: nat \Rightarrow nat$

**lemma** *Example1*:
 $\|- \{\!|True|\!\}$
   *COBEGIN SCHEME* $[0 \leq i < n]$ $\{\!|True|\!\}$ $\acute{}a := \acute{}a\ (i := 0)$ $\{\!|\acute{}a\ i = 0|\!\}$ *COEND*
  $\{\!|\forall i < n.\ \acute{}a\ i = 0|\!\}$
⟨*proof*⟩

Same example with lists as auxiliary variables.

**record** *Example1-list* =
  $A :: nat\ list$
**lemma** *Example1-list*:
 $\|- \{\!|n < length\ \acute{}A|\!\}$
   *COBEGIN*
     *SCHEME* $[0 \leq i < n]$ $\{\!|n < length\ \acute{}A|\!\}$ $\acute{}A := \acute{}A[i := 0]$ $\{\!|\acute{}A!i = 0|\!\}$
   *COEND*
    $\{\!|\forall i < n.\ \acute{}A!i = 0|\!\}$
⟨*proof*⟩

#### Increment a Variable in Parallel

First some lemmas about summation properties.

**lemma** *Example2-lemma2-aux*: $!!b.\ j < n \Longrightarrow$
 $(\sum i = 0..<n.\ (b\ i :: nat)) =$
 $(\sum i = 0..<j.\ b\ i) + b\ j + (\sum i = 0..<n - (Suc\ j)\ .\ b\ (Suc\ j + i))$
⟨*proof*⟩

**lemma** *Example2-lemma2-aux2*:
  $!!b.\ j \leq s \Longrightarrow (\sum i :: nat = 0..<j.\ (b\ (s := t))\ i) = (\sum i = 0..<j.\ b\ i)$
⟨*proof*⟩

**lemma** *Example2-lemma2*:
 $!!b.\ [\![j < n;\ b\ j = 0]\!] \Longrightarrow Suc\ (\sum i :: nat = 0..<n.\ b\ i) = (\sum i = 0..<n.\ (b\ (j := Suc\ 0))$
$i)$
⟨*proof*⟩

**record** *Example2* =
 $c :: nat \Rightarrow nat$
 $x :: nat$

**lemma** *Example-2*: $0 < n \Longrightarrow$
 $\|- \{\!|\acute{}x = 0 \wedge (\sum i = 0..<n.\ \acute{}c\ i) = 0|\!\}$
 *COBEGIN*

$SCHEME\ [0{\leq}i{<}n]$
$\{\!|\ {}'x{=}(\sum i{=}0..{<}n.\ {}'c\ i) \wedge {}'c\ i{=}0\ |\!\}$
$\langle\ {}'x{:=}{}'x{+}(Suc\ 0),,\ {}'c{:=}{}'c\ (i{:=}(Suc\ 0))\ \rangle$
$\{\!|\ {}'x{=}(\sum i{=}0..{<}n.\ {}'c\ i) \wedge {}'c\ i{=}(Suc\ 0)\ |\!\}$
$COEND$
$\{\!|\ {}'x{=}n\ |\!\}$
$\langle proof \rangle$

**end**

# Chapter 2

# Case Study: Single and Multi-Mutator Garbage Collection Algorithms

## 2.1 Formalization of the Memory

**theory** *Graph* **imports** *Main* **begin**

**datatype** *node* = *Black* | *White*

**type-synonym** *nodes* = *node list*
**type-synonym** *edge* = *nat* × *nat*
**type-synonym** *edges* = *edge list*

**consts** *Roots* :: *nat set*

**definition** *Proper-Roots* :: *nodes* ⇒ *bool* **where**
*Proper-Roots M* ≡ *Roots*≠{} ∧ *Roots* ⊆ {*i*. *i*<*length M*}

**definition** *Proper-Edges* :: (*nodes* × *edges*) ⇒ *bool* **where**
*Proper-Edges* ≡ (λ(*M*,*E*). ∀ *i*<*length E*. *fst*(*E*!*i*)<*length M* ∧ *snd*(*E*!*i*)<*length M*)

**definition** *BtoW* :: (*edge* × *nodes*) ⇒ *bool* **where**
*BtoW* ≡ (λ(*e*,*M*). (*M*!*fst e*)=*Black* ∧ (*M*!*snd e*)≠*Black*)

**definition** *Blacks* :: *nodes* ⇒ *nat set* **where**
*Blacks M* ≡ {*i*. *i*<*length M* ∧ *M*!*i*=*Black*}

**definition** *Reach* :: *edges* ⇒ *nat set* **where**
*Reach E* ≡ {*x*. (∃ *path*. 1<*length path* ∧ *path*!(*length path* − 1)∈*Roots* ∧ *x*=*path*!0
∧ (∀ *i*<*length path* − 1. (∃ *j*<*length E*. *E*!*j*=(*path*!(*i*+1), *path*!*i*))))
∨ *x*∈*Roots*}

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

### 2.1.1 Proofs about Graphs

**lemmas** *Graph-defs= Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def*
**declare** *Graph-defs* [*simp*]

#### Graph 1

**lemma** *Graph1-aux* [*rule-format*]:
$[\![\ Roots{\subseteq}Blacks\ M;\ \forall\, i{<}length\ E.\ \neg BtoW(E!i,M)]\!]$
$\Longrightarrow 1{<}\ length\ path \longrightarrow (path!(length\ path\ -\ 1)){\in}Roots \longrightarrow$
$(\forall\, i{<}length\ path\ -\ 1.\ (\exists\, j.\ j\ <\ length\ E \wedge E!j{=}(path!(Suc\ i),\ path!i)))$
$\longrightarrow M!(path!0) = Black$
⟨*proof*⟩

**lemma** *Graph1*:
$[\![Roots{\subseteq}Blacks\ M;\ Proper\text{-}Edges(M,\ E);\ \forall\, i{<}length\ E.\ \neg BtoW(E!i,M)\ ]\!]$
$\Longrightarrow Reach\ E{\subseteq}Blacks\ M$
⟨*proof*⟩

#### Graph 2

**lemma** *Ex-first-occurrence* [*rule-format*]:
$P\ (n{::}nat) \longrightarrow (\exists\, m.\ P\ m \wedge (\forall\, i.\ i{<}m \longrightarrow \neg\ P\ i))$
⟨*proof*⟩

**lemma** *Compl-lemma*: $(n{::}nat){\leq}l \Longrightarrow (\exists\, m.\ m{\leq}l \wedge n{=}l\ -\ m)$
⟨*proof*⟩

**lemma** *Ex-last-occurrence*:
$[\![P\ (n{::}nat);\ n{\leq}l]\!] \Longrightarrow (\exists\, m.\ P\ (l\ -\ m) \wedge (\forall\, i.\ i{<}m \longrightarrow \neg P\ (l\ -\ i)))$
⟨*proof*⟩

**lemma** *Graph2*:
$[\![T \in Reach\ E;\ R{<}length\ E]\!] \Longrightarrow T \in Reach\ (E[R{:=}(fst(E!R),\ T)])$
⟨*proof*⟩

#### Graph 3

**declare** *min.absorb1* [*simp*] *min.absorb2* [*simp*]

**lemma** *Graph3*:
$[\![\ T{\in}Reach\ E;\ R{<}length\ E\ ]\!] \Longrightarrow Reach(E[R{:=}(fst(E!R),T)]) \subseteq Reach\ E$
⟨*proof*⟩

### Graph 4

**lemma** *Graph4*:
$[\![T \in Reach\ E;\ Roots{\subseteq}Blacks\ M;\ I{\leq}length\ E;\ T{<}length\ M;\ R{<}length\ E;$
$\forall i{<}I.\ \neg BtoW(E!i,M);\ R{<}I;\ M!fst(E!R){=}Black;\ M!T{\neq}Black]\!] \Longrightarrow$
$(\exists r.\ I{\leq}r \wedge r{<}length\ E \wedge BtoW(E[R{:=}(fst(E!R),T)]!r,M))$
⟨*proof*⟩

**declare** *min.absorb1* [*simp del*] *min.absorb2* [*simp del*]

### Graph 5

**lemma** *Graph5*:
$[\![\ T \in Reach\ E\ ;\ Roots \subseteq Blacks\ M;\ \forall i{<}R.\ \neg BtoW(E!i,M);\ T{<}length\ M;$
$R{<}length\ E;\ M!fst(E!R){=}Black;\ M!snd(E!R){=}Black;\ M!T \neq Black]\!]$
$\Longrightarrow (\exists r.\ R{<}r \wedge r{<}length\ E \wedge BtoW(E[R{:=}(fst(E!R),T)]!r,M))$
⟨*proof*⟩

### Other lemmas about graphs

**lemma** *Graph6*:
$[\![Proper\text{-}Edges(M,E);\ R{<}length\ E\ ;\ T{<}length\ M]\!] \Longrightarrow Proper\text{-}Edges(M,E[R{:=}(fst(E!R),T)])$
⟨*proof*⟩

**lemma** *Graph7*:
$[\![Proper\text{-}Edges(M,E)]\!] \Longrightarrow Proper\text{-}Edges(M[T{:=}a],E)$
⟨*proof*⟩

**lemma** *Graph8*:
$[\![Proper\text{-}Roots(M)]\!] \Longrightarrow Proper\text{-}Roots(M[T{:=}a])$
⟨*proof*⟩

Some specific lemmata for the verification of garbage collection algorithms.

**lemma** *Graph9*: $j{<}length\ M \Longrightarrow Blacks\ M{\subseteq}Blacks\ (M[j := Black])$
⟨*proof*⟩

**lemma** *Graph10* [*rule-format* (*no-asm*)]: $\forall i.\ M!i{=}a \longrightarrow M[i{:=}a]{=}M$
⟨*proof*⟩

**lemma** *Graph11* [*rule-format* (*no-asm*)]:
$[\![\ M!j{\neq}Black;j{<}length\ M]\!] \Longrightarrow Blacks\ M \subset Blacks\ (M[j := Black])$
⟨*proof*⟩

**lemma** *Graph12*: $[\![a{\subseteq}Blacks\ M;j{<}length\ M]\!] \Longrightarrow a{\subseteq}Blacks\ (M[j := Black])$
⟨*proof*⟩

**lemma** *Graph13*: $[\![a{\subset}\ Blacks\ M;j{<}length\ M]\!] \Longrightarrow a \subset Blacks\ (M[j := Black])$
⟨*proof*⟩

**declare** *Graph-defs* [*simp del*]

**end**

## 2.2 The Single Mutator Case

**theory** *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

**declare** *psubsetE* [*rule del*]

Declaration of variables:

**record** *gar-coll-state* =
  *M* :: *nodes*
  *E* :: *edges*
  *bc* :: *nat set*
  *obc* :: *nat set*
  *Ma* :: *nodes*
  *ind* :: *nat*
  *k* :: *nat*
  *z* :: *bool*

### 2.2.1 The Mutator

The mutator first redirects an arbitrary edge $R$ from an arbitrary accessible node towards an arbitrary accessible node $T$. It then colors the new target $T$ black.

We declare the arbitrarily selected node and edge as constants:

**consts** *R* :: *nat*  *T* :: *nat*

The following predicate states, given a list of nodes $m$ and a list of edges $e$, the conditions under which the selected edge $R$ and node $T$ are valid:

**definition** *Mut-init* :: *gar-coll-state* ⇒ *bool* **where**
  *Mut-init* ≡ « *T* ∈ *Reach* ´*E* ∧ *R* < *length* ´*E* ∧ *T* < *length* ´*M* »

For the mutator we consider two modules, one for each action. An auxiliary variable ´*z* is set to false if the mutator has already redirected an edge but has not yet colored the new target.

**definition** *Redirect-Edge* :: *gar-coll-state ann-com* **where**
  *Redirect-Edge* ≡ {|´*Mut-init* ∧ ´*z*|} ⟨´*E*:=´*E*[*R*:=(*fst*(´*E*!*R*), *T*)],, ´*z*:= (¬´*z*)⟩

**definition** *Color-Target* :: *gar-coll-state ann-com* **where**
  *Color-Target* ≡ {|´*Mut-init* ∧ ¬´*z*|} ⟨´*M*:=´*M*[*T*:=*Black*],, ´*z*:= (¬´*z*)⟩

**definition** *Mutator* :: *gar-coll-state ann-com* **where**
  *Mutator* ≡
  {|´*Mut-init* ∧ ´*z*|}
  *WHILE True INV* {|´*Mut-init* ∧ ´*z*|}
  *DO  Redirect-Edge* ;; *Color-Target  OD*

### Correctness of the mutator

**lemmas** *mutator-defs = Mut-init-def Redirect-Edge-def Color-Target-def*

**lemma** *Redirect-Edge*:
⊢ *Redirect-Edge pre*(*Color-Target*)
⟨*proof*⟩

**lemma** *Color-Target*:
⊢ *Color-Target* {| ´*Mut-init* ∧ ´*z* |}
⟨*proof*⟩

**lemma** *Mutator*:
⊢ *Mutator* {|*False*|}
⟨*proof*⟩

## 2.2.2 The Collector

A constant *M-init* is used to give ´*Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

**consts** *M-init* :: *nodes*

**definition** *Proper-M-init* :: *gar-coll-state* ⇒ *bool* **where**
*Proper-M-init* ≡ « *Blacks M-init=Roots* ∧ *length M-init=length* ´*M* »

**definition** *Proper* :: *gar-coll-state* ⇒ *bool* **where**
*Proper* ≡ « *Proper-Roots* ´*M* ∧ *Proper-Edges*(´*M*, ´*E*) ∧ ´*Proper-M-init* »

**definition** *Safe* :: *gar-coll-state* ⇒ *bool* **where**
*Safe* ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

**lemmas** *collector-defs = Proper-M-init-def Proper-def Safe-def*

### Blackening the roots

**definition** *Blacken-Roots* :: *gar-coll-state ann-com* **where**
*Blacken-Roots* ≡
{| ´*Proper* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ ´*ind=0* |}
*WHILE* ´*ind*<*length* ´*M*
*INV* {| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M* |}
*DO* {| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}
*IF* ´*ind*∈*Roots THEN*
{| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots* |}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{| ´*Proper* ∧ (∀ *i*<´*ind*+*1*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}
´*ind*:=´*ind*+*1*

*OD*

**lemma** *Blacken-Roots*:
$\vdash$ *Blacken-Roots* {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* |}
⟨*proof*⟩

### Propagating black

**definition** *PBInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
*PBInv* ≡ « λ*ind*. ´*obc* < *Blacks* ´*M* ∨ (∀ *i* <*ind*. ¬*BtoW* (´*E*!*i*, ´*M*) ∨
(¬´*z* ∧ *i*=*R* ∧ (*snd*(´*E*!*R*)) = *T* ∧ (∃ *r*. *ind* ≤ *r* ∧ *r* < *length* ´*E* ∧ *BtoW*(´*E*!*r*,´*M*))))»

**definition** *Propagate-Black-aux* :: *gar-coll-state ann-com* **where**
*Propagate-Black-aux* ≡
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* ∧ ´*ind*=*0* |}
*WHILE* ´*ind*<*length* ´*E*
*INV* {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*≤*length* ´*E* |}
*DO* {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* |}
*IF* ´*M*!(*fst* (´*E*!´*ind*)) = *Black THEN*
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black* |}
´*M*:=´*M*[*snd*(´*E*!´*ind*):=*Black*];;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* (´*ind* + *1*) ∧ ´*ind*<*length* ´*E* |}
´*ind*:=´*ind*+*1*
*FI*
*OD*

**lemma** *Propagate-Black-aux*:
$\vdash$ *Propagate-Black-aux*
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ( ´*obc* < *Blacks* ´*M* ∨ ´*Safe*) |}
⟨*proof*⟩

### Refining propagating black

**definition** *Auxk* :: *gar-coll-state* ⇒ *bool* **where**
*Auxk* ≡ «´*k*<*length* ´*M* ∧ (´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨
´*obc*<*Blacks* ´*M* ∨ (¬´*z* ∧ ´*ind*=*R* ∧ *snd*(´*E*!*R*)=*T*
∧ (∃ *r*. ´*ind*<*r* ∧ *r*<*length* ´*E* ∧ *BtoW*(´*E*!*r*, ´*M*))))»

**definition** *Propagate-Black* :: *gar-coll-state ann-com* **where**
*Propagate-Black* ≡
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* ∧ ´*ind*=*0* |}

*WHILE ´ind<length ´E*
*INV {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ ´PBInv ´ind ∧ ´ind≤length ´E|}*
*DO {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ ´PBInv ´ind ∧ ´ind<length ´E|}*
*IF (´M!(fst (´E!´ind)))=Black THEN*
*{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black|}*
*´k:=(snd(´E!´ind));;*
*{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black*
*∧ ´Auxk|}*
*⟨´M:=´M[´k:=Black],, ´ind:=´ind+1⟩*
*ELSE {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ ´PBInv ´ind ∧ ´ind<length ´E|}*
*⟨IF (´M!(fst (´E!´ind)))≠Black THEN ´ind:=´ind+1 FI⟩*
*FI*
*OD*

**lemma** *Propagate-Black*:
*⊢ Propagate-Black*
*{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ (´obc < Blacks ´M ∨ ´Safe)|}*
⟨*proof*⟩

## Counting black nodes

**definition** *CountInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
*CountInv* ≡ « λ*ind*. {*i*. *i*<*ind* ∧ ´*Ma*!*i*=*Black*}⊆´*bc* »

**definition** *Count* :: *gar-coll-state ann-com* **where**
*Count* ≡
*{|´Proper ∧ Roots⊆Blacks ´M*
*∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}|}*
*´ind:=0;;*
*{|´Proper ∧ Roots⊆Blacks ´M*
*∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}*
*∧ ´ind=0|}*
*WHILE ´ind<length ´M*
*INV {|´Proper ∧ Roots⊆Blacks ´M*
*∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ length ´Ma=length ´M ∧ ´CountInv ´ind*
*∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind≤length ´M|}*
*DO {|´Proper ∧ Roots⊆Blacks ´M*
*∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M*
*∧ length ´Ma=length ´M ∧ ´CountInv ´ind*
*∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}*

```
    IF ´M!´ind=Black
      THEN {| ´Proper ∧ Roots⊆Blacks ´M
            ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
            ∧ length ´Ma=length ´M ∧ ´CountInv ´ind
          ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
      ´bc:=insert ´ind ´bc
    FI;;
    {| ´Proper ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ length ´Ma=length ´M ∧ ´CountInv (´ind+1)
     ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}
    ´ind:=´ind+1
  OD
```

**lemma** *Count*:
```
 ⊢ Count
 {| ´Proper ∧ Roots⊆Blacks ´M
  ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆´bc ∧ ´bc⊆Blacks ´M ∧ length ´Ma=length
´M
  ∧ (´obc < Blacks ´Ma ∨ ´Safe)|}
⟨proof⟩
```

### Appending garbage nodes to the free list

**axiomatization** *Append-to-free* :: *nat* × *edges* ⇒ *edges*
**where**
```
 Append-to-free0: length (Append-to-free (i, e)) = length e and
 Append-to-free1: Proper-Edges (m, e)
                ⟹ Proper-Edges (m, Append-to-free(i, e)) and
 Append-to-free2: i ∉ Reach e
   ⟹ n ∈ Reach (Append-to-free(i, e)) = ( n = i ∨ n ∈ Reach e)
```

**definition** *AppendInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
```
 AppendInv ≡ «λind. ∀ i<length ´M. ind≤i ⟶ i∈Reach ´E ⟶ ´M!i=Black»
```

**definition** *Append* :: *gar-coll-state ann-com* **where**
```
  Append ≡
 {| ´Proper ∧ Roots⊆Blacks ´M ∧ ´Safe|}
 ´ind:=0;;
 {| ´Proper ∧ Roots⊆Blacks ´M ∧ ´Safe ∧ ´ind=0|}
   WHILE ´ind<length ´M
    INV {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind≤length ´M|}
   DO {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M|}
     IF ´M!´ind=Black THEN
       {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
       ´M:=´M[´ind:=White]
     ELSE {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M ∧ ´ind∉Reach ´E|}
          ´E:=Append-to-free(´ind,´E)
     FI;;
```

$\{\!|\ \acute{}Proper \wedge \acute{}AppendInv\ (\acute{}ind{+}1) \wedge \acute{}ind{<}length\ \acute{}M\ |\!\}$
$\acute{}ind{:=}\acute{}ind{+}1$
*OD*

**lemma** *Append*:
$\vdash$ *Append* $\{\!|\ \acute{}Proper |\!\}$
⟨*proof*⟩

### Correctness of the Collector

**definition** *Collector* :: *gar-coll-state ann-com* **where**
*Collector* ≡
$\{\!|\ \acute{}Proper |\!\}$
*WHILE True INV* $\{\!|\ \acute{}Proper |\!\}$
*DO*
*Blacken-Roots*;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M |\!\}$
$\acute{}obc{:=}\{\}$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} |\!\}$
$\acute{}bc{:=}Roots$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} \wedge \acute{}bc{=}Roots |\!\}$
$\acute{}Ma{:=}M\text{-}init$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} \wedge \acute{}bc{=}Roots \wedge \acute{}Ma{=}M\text{-}init |\!\}$
*WHILE* $\acute{}obc{\neq}\acute{}bc$
*INV* $\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M$
$\wedge\ \acute{}obc{\subseteq}Blacks\ \acute{}Ma \wedge Blacks\ \acute{}Ma{\subseteq}\acute{}bc \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M$
$\wedge\ length\ \acute{}Ma{=}length\ \acute{}M \wedge (\acute{}obc < Blacks\ \acute{}Ma \vee \acute{}Safe) |\!\}$
*DO* $\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M |\!\}$
$\acute{}obc{:=}\acute{}bc$;;
*Propagate-Black*;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M$
$\wedge\ (\acute{}obc < Blacks\ \acute{}M \vee \acute{}Safe) |\!\}$
$\acute{}Ma{:=}\acute{}M$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{\subseteq}Blacks\ \acute{}Ma$
$\wedge\ Blacks\ \acute{}Ma{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M \wedge length\ \acute{}Ma{=}length\ \acute{}M$
$\wedge\ (\acute{}obc < Blacks\ \acute{}Ma \vee \acute{}Safe) |\!\}$
$\acute{}bc{:=}\{\}$;;
*Count*
*OD*;;
*Append*
*OD*

**lemma** *Collector*:
$\vdash$ *Collector* $\{\!|False|\!\}$
⟨*proof*⟩

## 2.2.3 Interference Freedom

**lemmas** *modules* = *Redirect-Edge-def Color-Target-def Blacken-Roots-def*
*Propagate-Black-def Count-def Append-def*

**lemmas** *Invariants = PBInv-def Auxk-def CountInv-def AppendInv-def*
**lemmas** *abbrev = collector-defs mutator-defs Invariants*

**lemma** *interfree-Blacken-Roots-Redirect-Edge*:
*interfree-aux* (*Some Blacken-Roots*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

**lemma** *interfree-Redirect-Edge-Blacken-Roots*:
*interfree-aux* (*Some Redirect-Edge*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

**lemma** *interfree-Blacken-Roots-Color-Target*:
*interfree-aux* (*Some Blacken-Roots*, {}, *Some Color-Target*)
⟨*proof*⟩

**lemma** *interfree-Color-Target-Blacken-Roots*:
*interfree-aux* (*Some Color-Target*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

**lemma** *interfree-Propagate-Black-Redirect-Edge*:
*interfree-aux* (*Some Propagate-Black*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

**lemma** *interfree-Redirect-Edge-Propagate-Black*:
*interfree-aux* (*Some Redirect-Edge*, {}, *Some Propagate-Black*)
⟨*proof*⟩

**lemma** *interfree-Propagate-Black-Color-Target*:
*interfree-aux* (*Some Propagate-Black*, {}, *Some Color-Target*)
⟨*proof*⟩

**lemma** *interfree-Color-Target-Propagate-Black*:
*interfree-aux* (*Some Color-Target*, {}, *Some Propagate-Black*)
⟨*proof*⟩

**lemma** *interfree-Count-Redirect-Edge*:
*interfree-aux* (*Some Count*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

**lemma** *interfree-Redirect-Edge-Count*:
*interfree-aux* (*Some Redirect-Edge*, {}, *Some Count*)
⟨*proof*⟩

**lemma** *interfree-Count-Color-Target*:
*interfree-aux* (*Some Count*, {}, *Some Color-Target*)
⟨*proof*⟩

**lemma** *interfree-Color-Target-Count*:
*interfree-aux* (*Some Color-Target*, {}, *Some Count*)

⟨*proof*⟩

**lemma** *interfree-Append-Redirect-Edge*:
*interfree-aux* (*Some Append*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

**lemma** *interfree-Redirect-Edge-Append*:
*interfree-aux* (*Some Redirect-Edge*, {}, *Some Append*)
⟨*proof*⟩

**lemma** *interfree-Append-Color-Target*:
*interfree-aux* (*Some Append*, {}, *Some Color-Target*)
⟨*proof*⟩

**lemma** *interfree-Color-Target-Append*:
*interfree-aux* (*Some Color-Target*, {}, *Some Append*)
⟨*proof*⟩

**lemmas** *collector-mutator-interfree* =
*interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target*
*interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target*
*interfree-Count-Redirect-Edge interfree-Count-Color-Target*
*interfree-Append-Redirect-Edge interfree-Append-Color-Target*
*interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots*
*interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black*
*interfree-Redirect-Edge-Count interfree-Color-Target-Count*
*interfree-Redirect-Edge-Append interfree-Color-Target-Append*

### Interference freedom Collector-Mutator

**lemma** *interfree-Collector-Mutator*:
*interfree-aux* (*Some Collector*, {}, *Some Mutator*)
⟨*proof*⟩

### Interference freedom Mutator-Collector

**lemma** *interfree-Mutator-Collector*:
*interfree-aux* (*Some Mutator*, {}, *Some Collector*)
⟨*proof*⟩

### The Garbage Collection algorithm

In total there are 289 verification conditions.

**lemma** *Gar-Coll*:
∥− {| ´*Proper* ∧ ´*Mut-init* ∧ ´*z* |}
*COBEGIN*
*Collector*
{| *False* |}
∥

*Mutator*
{|*False*|}
*COEND*
{|*False*|}
⟨*proof*⟩

**end**

## 2.3 The Multi-Mutator Case

**theory** *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

**record** *mut* =
*Z* :: *bool*
*R* :: *nat*
*T* :: *nat*

Declaration of variables:

**record** *mul-gar-coll-state* =
*M* :: *nodes*
*E* :: *edges*
*bc* :: *nat set*
*obc* :: *nat set*
*Ma* :: *nodes*
*ind* :: *nat*
*k* :: *nat*
*q* :: *nat*
*l* :: *nat*
*Muts* :: *mut list*

### 2.3.1 The Mutators

**definition** *Mul-mut-init* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
*Mul-mut-init* ≡ « λ*n*. *n*=*length* ´*Muts* ∧ (∀ *i*<*n*. *R* (´*Muts*!*i*)<*length* ´*E*
∧ *T* (´*Muts*!*i*)<*length* ´*M*) »

**definition** *Mul-Redirect-Edge* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
*Mul-Redirect-Edge j n* ≡
{|´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)|}
⟨*IF T*(´*Muts*!*j*) ∈ *Reach* ´*E THEN*
´*E*:= ´*E*[*R* (´*Muts*!*j*):= (*fst* (´*E*!*R*(´*Muts*!*j*)), *T* (´*Muts*!*j*))] *FI*,,
´*Muts*:= ´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*False*|)]⟩

**definition** *Mul-Color-Target* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
*Mul-Color-Target j n* ≡
{|´*Mul-mut-init n* ∧ ¬ *Z* (´*Muts*!*j*)|}
⟨´*M*:=´*M*[*T* (´*Muts*!*j*):=*Black*],, ´*Muts*:=´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*True*|)]⟩

**definition** *Mul-Mutator* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
*Mul-Mutator j n* ≡
{| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
*WHILE True*
*INV* {| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
*DO Mul-Redirect-Edge j n* ;;
*Mul-Color-Target j n*
*OD*

**lemmas** *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

### Correctness of the proof outline of one mutator

**lemma** *Mul-Redirect-Edge*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Redirect-Edge j n*
*pre*(*Mul-Color-Target j n*)
⟨*proof*⟩

**lemma** *Mul-Color-Target*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Color-Target j n*
{| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
⟨*proof*⟩

**lemma** *Mul-Mutator*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Mutator j n* {|*False*|}
⟨*proof*⟩

### Interference freedom between mutators

**lemma** *Mul-interfree-Redirect-Edge-Redirect-Edge*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
*interfree-aux* (*Some* (*Mul-Redirect-Edge i n*),{}, *Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Redirect-Edge-Color-Target*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
*interfree-aux* (*Some*(*Mul-Redirect-Edge i n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Redirect-Edge*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
*interfree-aux* (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Color-Target*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
*interfree-aux* (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

**lemmas** *mul-mutator-interfree* =
*Mul-interfree-Redirect-Edge-Redirect-Edge Mul-interfree-Redirect-Edge-Color-Target*
*Mul-interfree-Color-Target-Redirect-Edge Mul-interfree-Color-Target-Color-Target*

**lemma** *Mul-interfree-Mutator-Mutator*: $[\![i < n; j < n; i \neq j]\!] \Longrightarrow$
*interfree-aux* (*Some* (*Mul-Mutator i n*), {}, *Some* (*Mul-Mutator j n*))
⟨*proof*⟩

### Modular Parameterized Mutators

**lemma** *Mul-Parameterized-Mutators*: $0<n \Longrightarrow$
$\|-$ {| ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*)) |}
*COBEGIN*
*SCHEME* [0≤ *j*< *n*]
*Mul-Mutator j n*
{|*False*|}
*COEND*
{|*False*|}
⟨*proof*⟩

## 2.3.2 The Collector

**definition** *Queue* :: *mul-gar-coll-state* ⇒ *nat* **where**
*Queue* ≡ « *length* (*filter* (λ*i*. ¬ *Z i* ∧ ´*M*!(*T i*) ≠ *Black*) ´*Muts*) »

**consts** *M-init* :: *nodes*

**definition** *Proper-M-init* :: *mul-gar-coll-state* ⇒ *bool* **where**
*Proper-M-init* ≡ « *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* »

**definition** *Mul-Proper* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
*Mul-Proper* ≡ « λ*n*. *Proper-Roots* ´*M* ∧ *Proper-Edges* (´*M*, ´*E*) ∧ ´*Proper-M-init*
∧ *n*=*length* ´*Muts* »

**definition** *Safe* :: *mul-gar-coll-state* ⇒ *bool* **where**
*Safe* ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

**lemmas** *mul-collector-defs* = *Proper-M-init-def Mul-Proper-def Safe-def*

### Blackening Roots

**definition** *Mul-Blacken-Roots* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
*Mul-Blacken-Roots n* ≡
{| ´*Mul-Proper n* |}
´*ind*:=*0*;;
{| ´*Mul-Proper n* ∧ ´*ind*=*0* |}
*WHILE* ´*ind*<*length* ´*M*
*INV* {| ´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length*
´*M* |}
*DO* {| ´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}

*IF ´ind∈Roots THEN*
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*|}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*+*1*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
´*ind*:=´*ind*+*1*
*OD*

**lemma** *Mul-Blacken-Roots*:
⊢ *Mul-Blacken-Roots n*
{|´*Mul-Proper n* ∧ *Roots* ⊆ *Blacks* ´*M*|}
⟨*proof*⟩

## Propagating Black

**definition** *Mul-PBInv* :: *mul-gar-coll-state* ⇒ *bool* **where**
*Mul-PBInv* ≡ «´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue*
∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*)) ∧ ´*l*≤´*Queue*»

**definition** *Mul-Auxk* :: *mul-gar-coll-state* ⇒ *bool* **where**
*Mul-Auxk* ≡ «´*l*<´*Queue* ∨ ´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨ ´*obc*⊂*Blacks* ´*M*»

**definition** *Mul-Propagate-Black* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
*Mul-Propagate-Black n* ≡
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*)|}
´*ind*:=*0*;;
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ *Blacks* ´*M*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*) ∧ ´*ind*=*0*|}
*WHILE ´ind*<*length ´E*
*INV* {|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ ´*ind*≤*length* ´*E*|}
*DO* {|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ ´*ind*<*length* ´*E*|}
*IF ´M*!(*fst* (´*E*!´*ind*))=*Black THEN*
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ (´*M*!*fst*(´*E*!´*ind*))=*Black* ∧ ´*ind*<*length* ´*E*|}
´*k*:=*snd*(´*E*!´*ind*);;
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue* ∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*))
∧ ´*l*≤´*Queue* ∧ ´*Mul-Auxk* ) ∧ ´*k*<*length* ´*M* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black*
∧ ´*ind*<*length* ´*E*|}

```
  ⟨´M:=´M[´k:=Black],,´ind:=´ind+1⟩
  ELSE {|´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ ´Mul-PBInv ∧ ´ind<length ´E|}
       ⟨IF ´M!(fst (´E!´ind))≠Black THEN ´ind:=´ind+1 FI⟩ FI
OD
```

**lemma** *Mul-Propagate-Black*:

```
⊢ Mul-Propagate-Black n
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue ∧ (´l≤´Queue ∨ ´obc⊂Blacks
´M))|}
⟨proof⟩
```

### Counting Black Nodes

**definition** *Mul-CountInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**

```
Mul-CountInv ≡ « λind. {i. i<ind ∧ ´Ma!i=Black}⊆´bc »
```

**definition** *Mul-Count* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**

```
Mul-Count n ≡
{|´Mul-Proper n ∧ Roots⊆Blacks ´M
 ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
 ∧ length ´Ma=length ´M
 ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
 ∧ ´q<n+1 ∧ ´bc={}|}
´ind:=0;;
{|´Mul-Proper n ∧ Roots⊆Blacks ´M
 ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
 ∧ length ´Ma=length ´M
 ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
 ∧ ´q<n+1 ∧ ´bc={} ∧ ´ind=0|}
WHILE ´ind<length ´M
  INV {|´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
       ∧ ´q<n+1 ∧ ´ind≤length ´M|}
DO {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M|}
  IF ´M!´ind=Black
  THEN {|´Mul-Proper n ∧ Roots⊆Blacks ´M
        ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
         ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
```

```
          ∧ ´q<n+1 ∧ ´ind<length ´M ∧ ´M!´ind=Black⦘
        ´bc:=insert ´ind ´bc
    FI;;
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv (´ind+1)
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M⦘
  ´ind:=´ind+1
  OD
```

**lemma** *Mul-Count*:
```
  ⊢ Mul-Count n
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1⦘
⟨proof⟩
```

### Appending garbage nodes to the free list

**axiomatization** *Append-to-free* :: *nat* × *edges* ⇒ *edges*
**where**
```
  Append-to-free0: length (Append-to-free (i, e)) = length e and
  Append-to-free1: Proper-Edges (m, e)
                   ⟹ Proper-Edges (m, Append-to-free(i, e)) and
  Append-to-free2: i ∉ Reach e
          ⟹ n ∈ Reach (Append-to-free(i, e)) = ( n = i ∨ n ∈ Reach e)
```

**definition** *Mul-AppendInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
```
  Mul-AppendInv ≡ « λind. (∀ i. ind≤i ⟶ i<length ´M ⟶ i∈Reach ´E ⟶
´M!i=Black)»
```

**definition** *Mul-Append* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
```
  Mul-Append n ≡
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe⦘
  ´ind:=0;;
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe ∧ ´ind=0⦘
  WHILE ´ind<length ´M
    INV ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind≤length ´M⦘
  DO ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M⦘
      IF ´M!´ind=Black THEN
    ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´M!´ind=Black⦘
      ´M:=´M[´ind:=White]
      ELSE
      ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´ind∉Reach
´E⦘
      ´E:=Append-to-free(´ind,´E)
```

```
    FI;;
  {´Mul-Proper n ∧ ´Mul-AppendInv (´ind+1) ∧ ´ind<length ´M}
   ´ind:=´ind+1
  OD

lemma Mul-Append:
  ⊢ Mul-Append n
     {´Mul-Proper n}
⟨proof⟩
```

## Collector

```
definition Mul-Collector :: nat ⇒  mul-gar-coll-state ann-com where
  Mul-Collector n ≡
{´Mul-Proper n}
WHILE True INV {´Mul-Proper n}
DO
Mul-Blacken-Roots n ;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M}
 ´obc:={};;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={}}
 ´bc:=Roots;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots}
 ´l:=0;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´l=0}
 WHILE ´l<n+1
   INV {´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧
        (´Safe ∨ (´l≤´Queue ∨ ´bc⊂Blacks ´M) ∧ ´l<n+1)}
 DO {´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´l≤´Queue ∨ ´bc⊂Blacks ´M)}
    ´obc:=´bc;;
    Mul-Propagate-Black n;;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
     ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M))}
    ´bc:={};;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
     ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M)) ∧ ´bc={}}
      ⟨ ´Ma:=´M,, ´q:=´Queue ⟩;;
    Mul-Count n;;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
     ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
     ∧ ´q<n+1}
    IF ´obc=´bc THEN
```

$\{\!|\ ´Mul\text{-}Proper\ n \wedge Roots{\subseteq}Blacks\ ´M$
$\wedge\ ´obc{\subseteq}Blacks\ ´Ma \wedge Blacks\ ´Ma{\subseteq}Blacks\ ´M \wedge ´bc{\subseteq}Blacks\ ´M$
$\wedge\ length\ ´Ma{=}length\ ´M \wedge Blacks\ ´Ma{\subseteq}´bc$
$\wedge\ (´Safe \vee ´obc{\subset}Blacks\ ´Ma \vee ´l{<}´q \wedge (´q{\leq}´Queue \vee ´obc{\subset}Blacks\ ´M))$
$\wedge\ ´q{<}n{+}1 \wedge ´obc{=}´bc\ |\!\}$
$´l{:=}´l{+}1$
$ELSE\ \{\!|\ ´Mul\text{-}Proper\ n \wedge Roots{\subseteq}Blacks\ ´M$
$\wedge\ ´obc{\subseteq}Blacks\ ´Ma \wedge Blacks\ ´Ma{\subseteq}Blacks\ ´M \wedge ´bc{\subseteq}Blacks\ ´M$
$\wedge\ length\ ´Ma{=}length\ ´M \wedge Blacks\ ´Ma{\subseteq}´bc$
$\wedge\ (´Safe \vee ´obc{\subset}Blacks\ ´Ma \vee ´l{<}´q \wedge (´q{\leq}´Queue \vee ´obc{\subset}Blacks\ ´M))$
$\wedge\ ´q{<}n{+}1 \wedge ´obc{\neq}´bc\ |\!\}$
$´l{:=}0\ FI$
$OD$;;
$Mul\text{-}Append\ n$
$OD$

**lemmas** *mul-modules = Mul-Redirect-Edge-def Mul-Color-Target-def Mul-Blacken-Roots-def Mul-Propagate-Black-def Mul-Count-def Mul-Append-def*

**lemma** *Mul-Collector*:
$\vdash Mul\text{-}Collector\ n$
$\{\!|False|\!\}$
⟨*proof*⟩

### 2.3.3 Interference Freedom

**lemma** *le-length-filter-update*[*rule-format*]:
$\forall i.\ (\neg P\ (list!i) \vee P\ j) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) \leq length(filter\ P\ (list[i{:=}j]))$
⟨*proof*⟩

**lemma** *less-length-filter-update* [*rule-format*]:
$\forall i.\ P\ j \wedge \neg(P\ (list!i)) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) < length(filter\ P\ (list[i{:=}j]))$
⟨*proof*⟩

**lemma** *Mul-interfree-Blacken-Roots-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!] \Longrightarrow$
*interfree-aux* (*Some*(*Mul-Blacken-Roots n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Redirect-Edge-Blacken-Roots*: $[\![0{\leq}j;\ j{<}n]\!]\Longrightarrow$
*interfree-aux* (*Some*(*Mul-Redirect-Edge j n* ),{},*Some* (*Mul-Blacken-Roots n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Blacken-Roots-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]\Longrightarrow$
*interfree-aux* (*Some*(*Mul-Blacken-Roots n*),{},*Some* (*Mul-Color-Target j n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Blacken-Roots*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Color-Target j n* ),{},*Some* (*Mul-Blacken-Roots n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Propagate-Black-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Redirect-Edge j n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Redirect-Edge-Propagate-Black*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Redirect-Edge j n* ),{},*Some* (*Mul-Propagate-Black n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Propagate-Black-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Color-Target j n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Propagate-Black*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Color-Target j n*),{},*Some*(*Mul-Propagate-Black n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Count-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Count n* ),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Redirect-Edge-Count*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Count n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Count-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Count n* ),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Count*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Count n* ))
⟨*proof*⟩

**lemma** *Mul-interfree-Append-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Redirect-Edge-Append*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Append n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Append-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
*interfree-aux* (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

**lemma** *Mul-interfree-Color-Target-Append*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$

*interfree-aux* (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Append n*))
⟨*proof*⟩

### Interference freedom Collector-Mutator

**lemmas** *mul-collector-mutator-interfree* =
*Mul-interfree-Blacken-Roots-Redirect-Edge Mul-interfree-Blacken-Roots-Color-Target*
*Mul-interfree-Propagate-Black-Redirect-Edge Mul-interfree-Propagate-Black-Color-Target*
*Mul-interfree-Count-Redirect-Edge Mul-interfree-Count-Color-Target*
*Mul-interfree-Append-Redirect-Edge Mul-interfree-Append-Color-Target*
*Mul-interfree-Redirect-Edge-Blacken-Roots Mul-interfree-Color-Target-Blacken-Roots*
*Mul-interfree-Redirect-Edge-Propagate-Black Mul-interfree-Color-Target-Propagate-Black*
*Mul-interfree-Redirect-Edge-Count Mul-interfree-Color-Target-Count*
*Mul-interfree-Redirect-Edge-Append Mul-interfree-Color-Target-Append*

**lemma** *Mul-interfree-Collector-Mutator*: *j*<*n* ⟹
*interfree-aux* (*Some* (*Mul-Collector n*), {}, *Some* (*Mul-Mutator j n*))
⟨*proof*⟩

### Interference freedom Mutator-Collector

**lemma** *Mul-interfree-Mutator-Collector*: *j* < *n* ⟹
*interfree-aux* (*Some* (*Mul-Mutator j n*), {}, *Some* (*Mul-Collector n*))
⟨*proof*⟩

### The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

**lemma** *Mul-Gar-Coll*:
⊨− {| ´*Mul-Proper n* ∧ ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))|}
*COBEGIN*
*Mul-Collector n*
{|*False*|}
‖
*SCHEME* [0≤ *j*< *n*]
*Mul-Mutator j n*
{|*False*|}
*COEND*
{|*False*|}
⟨*proof*⟩

**end**

# Chapter 3

# The Rely-Guarantee Method

## 3.1 Abstract Syntax

**theory** *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (bexp) as sets of states. Syntax of commands *com* and parallel commands *par-com*.

**type-synonym** *'a bexp* = *'a set*

**datatype** *'a com* =
  *Basic 'a ⇒'a*
| *Seq 'a com 'a com*
| *Cond 'a bexp 'a com 'a com*
| *While 'a bexp 'a com*
| *Await 'a bexp 'a com*

**type-synonym** *'a par-com* = *'a com option list*

**end**

## 3.2 Operational Semantics

**theory** *RG-Tran*
**imports** *RG-Com*
**begin**

### 3.2.1 Semantics of Component Programs

#### Environment transitions

**type-synonym** *'a conf* = (*('a com) option*) × *'a*

**inductive-set**
  *etran* :: (*'a conf* × *'a conf*) *set*
  **and** *etran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (‹- −*e*→ -› [*81*,*81*] *80*)

**where**
  $P\ -e\rightarrow\ Q \equiv (P,Q) \in etran$
| *Env*: $(P,\ s)\ -e\rightarrow\ (P,\ t)$

**lemma** *etranE*: $c\ -e\rightarrow\ c' \Longrightarrow (\bigwedge P\ s\ t.\ c = (P,\ s) \Longrightarrow c' = (P,\ t) \Longrightarrow Q) \Longrightarrow Q$
  ⟨*proof*⟩

### Component transitions

**inductive-set**
  *ctran* :: $('a\ conf \times 'a\ conf)\ set$
  **and** *ctran′* :: $'a\ conf \Rightarrow 'a\ conf \Rightarrow bool$  (‹- $-c\rightarrow$ -› [*81*,*81*] *80*)
  **and** *ctrans* :: $'a\ conf \Rightarrow 'a\ conf \Rightarrow bool$  (‹- $-c*\rightarrow$ -› [*81*,*81*] *80*)
**where**
  $P\ -c\rightarrow\ Q \equiv (P,Q) \in ctran$
| $P\ -c*\rightarrow\ Q \equiv (P,Q) \in ctran^*$

| *Basic*: $(Some(Basic\ f),\ s)\ -c\rightarrow\ (None,\ f\ s)$

| *Seq1*: $(Some\ P0,\ s)\ -c\rightarrow\ (None,\ t) \Longrightarrow (Some(Seq\ P0\ P1),\ s)\ -c\rightarrow\ (Some\ P1,\ t)$

| *Seq2*: $(Some\ P0,\ s)\ -c\rightarrow\ (Some\ P2,\ t) \Longrightarrow (Some(Seq\ P0\ P1),\ s)\ -c\rightarrow\ (Some(Seq\ P2\ P1),\ t)$

| *CondT*: $s\in b\ \Longrightarrow (Some(Cond\ b\ P1\ P2),\ s)\ -c\rightarrow\ (Some\ P1,\ s)$
| *CondF*: $s\notin b \Longrightarrow (Some(Cond\ b\ P1\ P2),\ s)\ -c\rightarrow\ (Some\ P2,\ s)$

| *WhileF*: $s\notin b \Longrightarrow (Some(While\ b\ P),\ s)\ -c\rightarrow\ (None,\ s)$
| *WhileT*: $s\in b\ \Longrightarrow (Some(While\ b\ P),\ s)\ -c\rightarrow\ (Some(Seq\ P\ (While\ b\ P)),\ s)$

| *Await*: $[\![s\in b;\ (Some\ P,\ s)\ -c*\rightarrow\ (None,\ t)]\!] \Longrightarrow (Some(Await\ b\ P),\ s)\ -c\rightarrow\ (None,\ t)$

**monos** *rtrancl-mono*

## 3.2.2 Semantics of Parallel Programs

**type-synonym** $'a\ par\text{-}conf = ('a\ par\text{-}com) \times 'a$

**inductive-set**
  *par-etran* :: $('a\ par\text{-}conf \times 'a\ par\text{-}conf)\ set$
  **and** *par-etran′* :: $['a\ par\text{-}conf,'a\ par\text{-}conf] \Rightarrow bool$ (‹- $-pe\rightarrow$ -› [*81*,*81*] *80*)
**where**
  $P\ -pe\rightarrow\ Q \equiv (P,Q) \in par\text{-}etran$
| *ParEnv*: $(Ps,\ s)\ -pe\rightarrow\ (Ps,\ t)$

**inductive-set**
  *par-ctran* :: $('a\ par\text{-}conf \times 'a\ par\text{-}conf)\ set$
  **and** *par-ctran′* :: $['a\ par\text{-}conf,'a\ par\text{-}conf] \Rightarrow bool$ (‹- $-pc\rightarrow$ -› [*81*,*81*] *80*)

**where**
  $P\ -pc\rightarrow\ Q \equiv (P,Q) \in$ *par-ctran*
| *ParComp*: $[\![i<$*length Ps*; $(Ps!i,\ s)\ -c\rightarrow\ (r,\ t)]\!] \Longrightarrow (Ps,\ s)\ -pc\rightarrow\ (Ps[i:=r],\ t)$

**lemma** *par-ctranE*: $c\ -pc\rightarrow\ c' \Longrightarrow$
  $(\bigwedge i\ Ps\ s\ r\ t.\ c = (Ps,\ s) \Longrightarrow c' = (Ps[i := r],\ t) \Longrightarrow i < length\ Ps \Longrightarrow$
    $(Ps\ !\ i,\ s)\ -c\rightarrow\ (r,\ t) \Longrightarrow P) \Longrightarrow P$
  ⟨*proof*⟩

### 3.2.3 Computations

#### Sequential computations

**type-synonym** *'a confs* = *'a conf list*

**inductive-set** *cptn* :: *'a confs set*
**where**
  *CptnOne*: $[(P,s)] \in$ *cptn*
| *CptnEnv*: $(P,\ t)\#xs \in$ *cptn* $\Longrightarrow (P,s)\#(P,t)\#xs \in$ *cptn*
| *CptnComp*: $[\![(P,s)\ -c\rightarrow\ (Q,t);\ (Q,\ t)\#xs \in$ *cptn* $]\!] \Longrightarrow (P,s)\#(Q,t)\#xs \in$ *cptn*

**definition** *cp* :: *('a com) option* ⇒ *'a* ⇒ *('a confs) set* **where**
  *cp P s* ≡ $\{l.\ l!0=(P,s) \wedge l \in$ *cptn*$\}$

#### Parallel computations

**type-synonym** *'a par-confs* = *'a par-conf list*

**inductive-set** *par-cptn* :: *'a par-confs set*
**where**
  *ParCptnOne*: $[(P,s)] \in$ *par-cptn*
| *ParCptnEnv*: $(P,t)\#xs \in$ *par-cptn* $\Longrightarrow (P,s)\#(P,t)\#xs \in$ *par-cptn*
| *ParCptnComp*: $[\![\ (P,s)\ -pc\rightarrow\ (Q,t);\ (Q,t)\#xs \in$ *par-cptn* $]\!] \Longrightarrow (P,s)\#(Q,t)\#xs$
$\in$ *par-cptn*

**definition** *par-cp* :: *'a par-com* ⇒ *'a* ⇒ *('a par-confs) set* **where**
  *par-cp P s* ≡ $\{l.\ l!0=(P,s) \wedge l \in$ *par-cptn*$\}$

### 3.2.4 Modular Definition of Computation

**definition** *lift* :: *'a com* ⇒ *'a conf* ⇒ *'a conf* **where**
  *lift Q* ≡ $\lambda(P,\ s).$ (*if P=None then* (*Some Q*,*s*) *else* (*Some*(*Seq* (*the P*) *Q*), *s*))

**inductive-set** *cptn-mod* :: *('a confs) set*
**where**
  *CptnModOne*: $[(P,\ s)] \in$ *cptn-mod*
| *CptnModEnv*: $(P,\ t)\#xs \in$ *cptn-mod* $\Longrightarrow (P,\ s)\#(P,\ t)\#xs \in$ *cptn-mod*
| *CptnModNone*: $[\![(Some\ P,\ s)\ -c\rightarrow\ (None,\ t);\ (None,\ t)\#xs \in$ *cptn-mod* $]\!] \Longrightarrow$
$(Some\ P,s)\#(None,\ t)\#xs \in$*cptn-mod*

| *CptnModCondT*: $[\![(Some\ P0,\ s)\#ys \in cptn\text{-}mod;\ s \in b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P0,\ s)\#ys \in cptn\text{-}mod$
| *CptnModCondF*: $[\![(Some\ P1,\ s)\#ys \in cptn\text{-}mod;\ s \notin b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P1,\ s)\#ys \in cptn\text{-}mod$
| *CptnModSeq1*: $[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ zs=map\ (lift\ P1)\ xs\ ]\!]$
$\Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$
| *CptnModSeq2*:
$[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P0,\ s)\#xs)) = None;$
$(Some\ P1,\ snd(last\ ((Some\ P0,\ s)\#xs)))\#ys \in cptn\text{-}mod;$
$zs=(map\ (lift\ P1)\ xs)@ys\ ]\!] \Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$

| *CptnModWhile1*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ s \in b;\ zs=map\ (lift\ (While\ b\ P))\ xs\ ]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$
| *CptnModWhile2*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P,\ s)\#xs))=None;\ s \in b;$
$zs=(map\ (lift\ (While\ b\ P))\ xs)@ys;$
$(Some(While\ b\ P),\ snd(last\ ((Some\ P,\ s)\#xs)))\#ys \in cptn\text{-}mod]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$

### 3.2.5 Equivalence of Both Definitions.

**lemma** *last-length*: $((a\#xs)!(length\ xs))=last\ (a\#xs)$
⟨*proof*⟩

**lemma** *div-seq* [*rule-format*]: $list \in cptn\text{-}mod \Longrightarrow$
$(\forall\ s\ P\ Q\ zs.\ list=(Some\ (Seq\ P\ Q),\ s)\#zs \longrightarrow$
$(\exists\ xs.\ (Some\ P,\ s)\#xs \in cptn\text{-}mod\ \wedge\ (zs=(map\ (lift\ Q)\ xs)\ \vee$
$(\ fst(((Some\ P,\ s)\#xs)!length\ xs)=None\ \wedge$
$(\exists\ ys.\ (Some\ Q,\ snd(((Some\ P,\ s)\#xs)!length\ xs))\#ys \in cptn\text{-}mod$
$\wedge\ zs=(map\ (lift\ (Q))\ xs)@ys)))))$
⟨*proof*⟩

**lemma** *cptn-onlyif-cptn-mod-aux* [*rule-format*]:
$\forall\ s\ Q\ t\ xs.((Some\ a,\ s),\ Q,\ t) \in ctran \longrightarrow (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
$\longrightarrow (Some\ a,\ s)\ \#\ (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
⟨*proof*⟩

**lemma** *cptn-onlyif-cptn-mod* [*rule-format*]: $c \in cptn \Longrightarrow c \in cptn\text{-}mod$
⟨*proof*⟩

**lemma** *lift-is-cptn*: $c{\in}cptn \Longrightarrow map\ (lift\ P)\ c \in cptn$
⟨*proof*⟩

**lemma** *cptn-append-is-cptn* [*rule-format*]:
$\forall\ b\ a.\ b\#c1{\in}cptn \longrightarrow\ a\#c2{\in}cptn \longrightarrow (b\#c1)!length\ c1=a \longrightarrow b\#c1@c2{\in}cptn$
⟨*proof*⟩

**lemma** *last-lift*: $[\![xs{\neq}[];\ fst(xs!(length\ xs - (Suc\ 0)))=None]\!]$

$\Longrightarrow$ *fst*((*map* (*lift P*) *xs*)!(*length* (*map* (*lift P*) *xs*)− (*Suc 0*)))=(*Some P*)
⟨*proof*⟩

**lemma** *last-fst* [*rule-format*]: *P*((*a*#*x*)!*length x*) ⟶ ¬*P a* ⟶ *P* (*x*!(*length x* − (*Suc 0*)))
⟨*proof*⟩

**lemma** *last-fst-esp*:
*fst*(((*Some a*,*s*)#*xs*)!(*length xs*))=*None* $\Longrightarrow$ *fst*(*xs*!(*length xs* − (*Suc 0*)))=*None*
⟨*proof*⟩

**lemma** *last-snd*: *xs*≠[] $\Longrightarrow$
*snd*(((*map* (*lift P*) *xs*))!(*length* (*map* (*lift P*) *xs*) − (*Suc 0*)))=*snd*(*xs*!(*length xs* − (*Suc 0*)))
⟨*proof*⟩

**lemma** *Cons-lift*: (*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)
⟨*proof*⟩

**lemma** *Cons-lift-append*:
(*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) @ *ys* = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)@ *ys*
⟨*proof*⟩

**lemma** *lift-nth*: *i*<*length xs* $\Longrightarrow$ *map* (*lift Q*) *xs* ! *i* = *lift Q* (*xs*! *i*)
⟨*proof*⟩

**lemma** *snd-lift*: *i*< *length xs* $\Longrightarrow$ *snd*(*lift Q* (*xs* ! *i*))= *snd* (*xs* ! *i*)
⟨*proof*⟩

**lemma** *cptn-if-cptn-mod*: *c* ∈ *cptn-mod* $\Longrightarrow$ *c* ∈ *cptn*
⟨*proof*⟩

**theorem** *cptn-iff-cptn-mod*: (*c* ∈ *cptn*) = (*c* ∈ *cptn-mod*)
⟨*proof*⟩

## 3.3 Validity of Correctness Formulas

### 3.3.1 Validity for Component Programs.

**type-synonym** *'a rgformula* =
*'a com* × *'a set* × (*'a* × *'a*) *set* × (*'a* × *'a*) *set* × *'a set*

**definition** *assum* :: (*'a set* × (*'a* × *'a*) *set*) ⇒ (*'a confs*) *set* **where**
*assum* ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
*c*!*i* −*e*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

**definition** *comm* :: ((*'a* × *'a*) *set* × *'a set*) ⇒ (*'a confs*) *set* **where**

*comm* ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
*c*!*i* −*c*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
(*fst* (*last c*) = *None* ⟶ *snd* (*last c*) ∈ *post*)}

**definition** *com-validity* :: 'a *com* ⇒ 'a *set* ⇒ ('a × 'a) *set* ⇒ ('a × 'a) *set* ⇒ 'a *set* ⇒ *bool*
(‹⊨ - *sat* [-, -, -, -]› [60,0,0,0,0] 45) **where**
⊨ *P sat* [*pre*, *rely*, *guar*, *post*] ≡
∀ *s*. *cp* (*Some P*) *s* ∩ *assum*(*pre*, *rely*) ⊆ *comm*(*guar*, *post*)

### 3.3.2 Validity for Parallel Programs.

**definition** *All-None* :: (('a *com*) *option*) *list* ⇒ *bool* **where**
*All-None xs* ≡ ∀ *c*∈*set xs*. *c*=*None*

**definition** *par-assum* :: ('a *set* × ('a × 'a) *set*) ⇒ ('a *par-confs*) *set* **where**
*par-assum* ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
*c*!*i* −*pe*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

**definition** *par-comm* :: (('a × 'a) *set* × 'a *set*) ⇒ ('a *par-confs*) *set* **where**
*par-comm* ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
*c*!*i* −*pc*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
(*All-None* (*fst* (*last c*)) ⟶ *snd*( *last c*) ∈ *post*)}

**definition** *par-com-validity* :: 'a *par-com* ⇒ 'a *set* ⇒ ('a × 'a) *set* ⇒ ('a × 'a) *set*
⇒ 'a *set* ⇒ *bool* (‹⊨ - *SAT* [-, -, -, -]› [60,0,0,0,0] 45) **where**
⊨ *Ps SAT* [*pre*, *rely*, *guar*, *post*] ≡
∀ *s*. *par-cp Ps s* ∩ *par-assum*(*pre*, *rely*) ⊆ *par-comm*(*guar*, *post*)

### 3.3.3 Compositionality of the Semantics

#### Definition of the conjoin operator

**definition** *same-length* :: 'a *par-confs* ⇒ ('a *confs*) *list* ⇒ *bool* **where**
*same-length c clist* ≡ (∀ *i*<*length clist*. *length*(*clist*!*i*)=*length c*)

**definition** *same-state* :: 'a *par-confs* ⇒ ('a *confs*) *list* ⇒ *bool* **where**
*same-state c clist* ≡ (∀ *i* <*length clist*. ∀ *j*<*length c*. *snd*(*c*!*j*) = *snd*((*clist*!*i*)!*j*))

**definition** *same-program* :: 'a *par-confs* ⇒ ('a *confs*) *list* ⇒ *bool* **where**
*same-program c clist* ≡ (∀ *j*<*length c*. *fst*(*c*!*j*) = *map* (λ*x*. *fst*(*nth x j*)) *clist*)

**definition** *compat-label* :: 'a *par-confs* ⇒ ('a *confs*) *list* ⇒ *bool* **where**
*compat-label c clist* ≡ (∀ *j*. *Suc j*<*length c* ⟶
(*c*!*j* −*pc*→ *c*!*Suc j* ∧ (∃ *i*<*length clist*. (*clist*!*i*)!*j* −*c*→ (*clist*!*i*)! *Suc j* ∧
(∀ *l*<*length clist*. *l*≠*i* ⟶ (*clist*!*l*)!*j* −*e*→ (*clist*!*l*)! *Suc j*))) ∨
(*c*!*j* −*pe*→ *c*!*Suc j* ∧ (∀ *i*<*length clist*. (*clist*!*i*)!*j* −*e*→ (*clist*!*i*)! *Suc j*)))

**definition** *conjoin* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* (‹- ∝ -› [*65*,*65*] *64*)
**where**
*c* ∝ *clist* ≡ (*same-length c clist*) ∧ (*same-state c clist*) ∧ (*same-program c clist*) ∧ (*compat-label c clist*)

### Some previous lemmas

**lemma** *list-eq-if* [*rule-format*]:
∀ *ys*. *xs*=*ys* ⟶ (*length xs* = *length ys*) ⟶ (∀ *i*<*length xs*. *xs*!*i*=*ys*!*i*)
⟨*proof*⟩

**lemma** *list-eq*: (*length xs* = *length ys* ∧ (∀ *i*<*length xs*. *xs*!*i*=*ys*!*i*)) = (*xs*=*ys*)
⟨*proof*⟩

**lemma** *nth-tl*: ⟦ *ys*!*0*=*a*; *ys*≠[] ⟧ ⟹ *ys*=(*a*#(*tl ys*))
⟨*proof*⟩

**lemma** *nth-tl-if* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P ys* ⟶ *P* (*a*#(*tl ys*))
⟨*proof*⟩

**lemma** *nth-tl-onlyif* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P* (*a*#(*tl ys*)) ⟶ *P ys*
⟨*proof*⟩

**lemma** *seq-not-eq1*: *Seq c1 c2*≠*c1*
⟨*proof*⟩

**lemma** *seq-not-eq2*: *Seq c1 c2*≠*c2*
⟨*proof*⟩

**lemma** *if-not-eq1*: *Cond b c1 c2* ≠*c1*
⟨*proof*⟩

**lemma** *if-not-eq2*: *Cond b c1 c2*≠*c2*
⟨*proof*⟩

**lemmas** *seq-and-if-not-eq* [*simp*] = *seq-not-eq1 seq-not-eq2*
*seq-not-eq1* [*THEN not-sym*] *seq-not-eq2* [*THEN not-sym*]
*if-not-eq1 if-not-eq2 if-not-eq1* [*THEN not-sym*] *if-not-eq2* [*THEN not-sym*]

**lemma** *prog-not-eq-in-ctran-aux*:
**assumes** *c*: (*P*,*s*) −*c*→ (*Q*,*t*)
**shows** *P*≠*Q* ⟨*proof*⟩

**lemma** *prog-not-eq-in-ctran* [*simp*]: ¬ (*P*,*s*) −*c*→ (*P*,*t*)
⟨*proof*⟩

**lemma** *prog-not-eq-in-par-ctran-aux* [*rule-format*]: (*P*,*s*) −*pc*→ (*Q*,*t*) ⟹ (*P*≠*Q*)
⟨*proof*⟩

**lemma** *prog-not-eq-in-par-ctran* [*simp*]: ¬ (*P*,*s*) −*pc*→ (*P*,*t*)
⟨*proof*⟩

**lemma** *tl-in-cptn*: ⟦ *a*#*xs* ∈*cptn*; *xs*≠[] ⟧ ⟹ *xs*∈*cptn*
⟨*proof*⟩

**lemma** *tl-zero*[*rule-format*]:
*P* (*ys*!*Suc j*) ⟶ *Suc j*<*length ys* ⟶ *ys*≠[] ⟶ *P* (*tl*(*ys*)!*j*)
⟨*proof*⟩

### 3.3.4 The Semantics is Compositional

**lemma** *aux-if* [*rule-format*]:
∀ *xs s clist*. (*length clist* = *length xs* ∧ (∀ *i*<*length xs*. (*xs*!*i*,*s*)#*clist*!*i* ∈ *cptn*)
∧ ((*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#*snd i*) (*zip xs clist*))
⟶ (*xs*, *s*)#*ys* ∈ *par-cptn*)
⟨*proof*⟩

**lemma** *aux-onlyif* [*rule-format*]: ∀ *xs s*. (*xs*, *s*)#*ys* ∈ *par-cptn* ⟶
(∃ *clist*. (*length clist* = *length xs*) ∧
(*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#(*snd i*)) (*zip xs clist*) ∧
(∀ *i*<*length xs*. (*xs*!*i*,*s*)#(*clist*!*i*) ∈ *cptn*))
⟨*proof*⟩

**lemma** *one-iff-aux*: *xs*≠[] ⟹ (∀ *ys*. ((*xs*, *s*)#*ys* ∈ *par-cptn*) =
(∃ *clist*. *length clist*= *length xs* ∧
((*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#(*snd i*)) (*zip xs clist*)) ∧
(∀ *i*<*length xs*. (*xs*!*i*,*s*)#(*clist*!*i*) ∈ *cptn*))) =
(*par-cp* (*xs*) *s* = {*c*. ∃ *clist*. (*length clist*)=(*length xs*) ∧
(∀ *i*<*length clist*. (*clist*!*i*) ∈ *cp*(*xs*!*i*) *s*) ∧ *c* ∝ *clist*})
⟨*proof*⟩

**theorem** *one*: *xs*≠[] ⟹
*par-cp xs s* = {*c*. ∃ *clist*. (*length clist*)=(*length xs*) ∧
(∀ *i*<*length clist*. (*clist*!*i*) ∈ *cp*(*xs*!*i*) *s*) ∧ *c* ∝ *clist*}
⟨*proof*⟩

**end**

## 3.4 The Proof System

**theory** *RG-Hoare* **imports** *RG-Tran* **begin**

### 3.4.1 Proof System for Component Programs

**declare** *Un-subset-iff* [*simp del*] *sup.bounded-iff* [*simp del*]

**definition** *stable* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *bool* **where**
*stable* ≡ λ*f g*. (∀ *x y*. *x* ∈ *f* ⟶ (*x*, *y*) ∈ *g* ⟶ *y* ∈ *f*)

**inductive**
*rghoare* :: [*'a com*, *'a set*, (*'a* × *'a*) *set*, (*'a* × *'a*) *set*, *'a set*] ⇒ *bool*
(‹⊢ - *sat* [-, -, -, -]› [*60*,*0*,*0*,*0*,*0] *45*)
**where**
*Basic*: ⟦ *pre* ⊆ {*s*. *f s* ∈ *post*}; {(*s*,*t*). *s* ∈ *pre* ∧ (*t*=*f s* ∨ *t*=*s*)} ⊆ *guar*;
*stable pre rely*; *stable post rely* ⟧
⟹ ⊢ *Basic f sat* [*pre*, *rely*, *guar*, *post*]

| *Seq*: ⟦ ⊢ *P sat* [*pre*, *rely*, *guar*, *mid*]; ⊢ *Q sat* [*mid*, *rely*, *guar*, *post*] ⟧
⟹ ⊢ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]

| *Cond*: ⟦ *stable pre rely*; ⊢ *P1 sat* [*pre* ∩ *b*, *rely*, *guar*, *post*];
⊢ *P2 sat* [*pre* ∩ −*b*, *rely*, *guar*, *post*]; ∀ *s*. (*s*,*s*)∈*guar* ⟧
⟹ ⊢ *Cond b P1 P2 sat* [*pre*, *rely*, *guar*, *post*]

| *While*: ⟦ *stable pre rely*; (*pre* ∩ −*b*) ⊆ *post*; *stable post rely*;
⊢ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*,*s*)∈*guar* ⟧
⟹ ⊢ *While b P sat* [*pre*, *rely*, *guar*, *post*]

| *Await*: ⟦ *stable pre rely*; *stable post rely*;
∀ *V*. ⊢ *P sat* [*pre* ∩ *b* ∩ {*V*}, {(*s*, *t*). *s* = *t*},
*UNIV*, {*s*. (*V*, *s*) ∈ *guar*} ∩ *post*] ⟧
⟹ ⊢ *Await b P sat* [*pre*, *rely*, *guar*, *post*]

| *Conseq*: ⟦ *pre* ⊆ *pre'*; *rely* ⊆ *rely'*; *guar'* ⊆ *guar*; *post'* ⊆ *post*;
⊢ *P sat* [*pre'*, *rely'*, *guar'*, *post'*] ⟧
⟹ ⊢ *P sat* [*pre*, *rely*, *guar*, *post*]

**definition** *Pre* :: *'a rgformula* ⇒ *'a set* **where**
*Pre x* ≡ *fst*(*snd x*)

**definition** *Post* :: *'a rgformula* ⇒ *'a set* **where**
*Post x* ≡ *snd*(*snd*(*snd*(*snd x*)))

**definition** *Rely* :: *'a rgformula* ⇒ (*'a* × *'a*) *set* **where**
*Rely x* ≡ *fst*(*snd*(*snd x*))

**definition** *Guar* :: *'a rgformula* ⇒ (*'a* × *'a*) *set* **where**
*Guar x* ≡ *fst*(*snd*(*snd*(*snd x*)))

**definition** *Com* :: *'a rgformula* ⇒ *'a com* **where**
*Com x* ≡ *fst x*

### 3.4.2 Proof System for Parallel Programs

**type-synonym** *'a par-rgformula* =
(*'a rgformula*) *list* × *'a set* × (*'a* × *'a*) *set* × (*'a* × *'a*) *set* × *'a set*

**inductive**
*par-rghoare* :: ('*a rgformula*) *list* ⇒ '*a set* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* ⇒ '*a set* ⇒ *bool*
(‹⊢ - *SAT* [-, -, -, -]› [*60*,*0*,*0*,*0*,*0*] *45*)
**where**
*Parallel*:
⟦ ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j*<*length xs* ∧ *j*≠*i*}. *Guar*(*xs*!*j*)) ⊆ *Rely*(*xs*!*i*);
(⋃*j*∈{*j*. *j*<*length xs*}. *Guar*(*xs*!*j*)) ⊆ *guar*;
*pre* ⊆ (⋂*i*∈{*i*. *i*<*length xs*}. *Pre*(*xs*!*i*));
(⋂*i*∈{*i*. *i*<*length xs*}. *Post*(*xs*!*i*)) ⊆ *post*;
∀ *i*<*length xs*. ⊢ *Com*(*xs*!*i*) *sat* [*Pre*(*xs*!*i*),*Rely*(*xs*!*i*),*Guar*(*xs*!*i*),*Post*(*xs*!*i*)] ⟧
⟹ ⊢ *xs SAT* [*pre*, *rely*, *guar*, *post*]

## 3.5 Soundness

### Some previous lemmas

**lemma** *tl-of-assum-in-assum*:
(*P*, *s*) # (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*) ⟹ *stable pre rely*
⟹ (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*)
⟨*proof*⟩

**lemma** *etran-in-comm*:
(*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*) ⟹ (*P*, *s*) # (*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*)
⟨*proof*⟩

**lemma** *ctran-in-comm*:
⟦(*s*, *s*) ∈ *guar*; (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)⟧
⟹ (*P*, *s*) # (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)
⟨*proof*⟩

**lemma** *takecptn-is-cptn* [*rule-format*, *elim*!]:
∀ *j*. *c* ∈ *cptn* ⟶ *take* (*Suc j*) *c* ∈ *cptn*
⟨*proof*⟩

**lemma** *dropcptn-is-cptn* [*rule-format*,*elim*!]:
∀ *j*<*length c*. *c* ∈ *cptn* ⟶ *drop j c* ∈ *cptn*
⟨*proof*⟩

**lemma** *takepar-cptn-is-par-cptn* [*rule-format*,*elim*]:
∀ *j*. *c* ∈ *par-cptn* ⟶ *take* (*Suc j*) *c* ∈ *par-cptn*
⟨*proof*⟩

**lemma** *droppar-cptn-is-par-cptn* [*rule-format*]:
∀ *j*<*length c*. *c* ∈ *par-cptn* ⟶ *drop j c* ∈ *par-cptn*
⟨*proof*⟩

**lemma** *tl-of-cptn-is-cptn*: ⟦*x* # *xs* ∈ *cptn*; *xs* ≠ []⟧ ⟹ *xs* ∈ *cptn*
⟨*proof*⟩

**lemma** *not-ctran-None* [*rule-format*]:
$\forall s.\ (None,\ s)\#xs \in cptn \longrightarrow (\forall i{<}length\ xs.\ ((None,\ s)\#xs)!i\ -e\rightarrow\ xs!i)$
⟨*proof*⟩

**lemma** *cptn-not-empty* [*simp*]:$[] \notin cptn$
⟨*proof*⟩

**lemma** *etran-or-ctran* [*rule-format*]:
$\forall m\ i.\ x{\in}cptn \longrightarrow m \leq length\ x$
$\longrightarrow (\forall i.\ Suc\ i < m \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i) \longrightarrow Suc\ i < m$
$\longrightarrow x!i\ -e\rightarrow\ x!Suc\ i$
⟨*proof*⟩

**lemma** *etran-or-ctran2* [*rule-format*]:
$\forall i.\ Suc\ i{<}length\ x \longrightarrow x{\in}cptn \longrightarrow (x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i)$
$\vee\ (x!i\ -e\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i)$
⟨*proof*⟩

**lemma** *etran-or-ctran2-disjI1*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -c\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i$
⟨*proof*⟩

**lemma** *etran-or-ctran2-disjI2*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -e\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i$
⟨*proof*⟩

**lemma** *not-ctran-None2* [*rule-format*]:
$[\![\ (None,\ s)\ \#\ xs\ {\in}cptn;\ i{<}length\ xs]\!] \Longrightarrow \neg\ ((None,\ s)\ \#\ xs)\ !\ i\ -c\rightarrow\ xs\ !\ i$
⟨*proof*⟩

**lemma** *Ex-first-occurrence* [*rule-format*]: $P\ (n{::}nat) \longrightarrow (\exists m.\ P\ m \wedge (\forall i{<}m.\ \neg\ P\ i))$
⟨*proof*⟩

**lemma** *stability* [*rule-format*]:
$\forall j\ k.\ x \in cptn \longrightarrow stable\ p\ rely \longrightarrow j{\leq}k \longrightarrow k{<}length\ x \longrightarrow snd(x!j){\in}p\ \longrightarrow$
$(\forall i.\ (Suc\ i){<}length\ x \longrightarrow$
$(x!i\ -e\rightarrow\ x!(Suc\ i)) \longrightarrow (snd(x!i),\ snd(x!(Suc\ i))) \in rely) \longrightarrow$
$(\forall i.\ j{\leq}i \wedge i{<}k \longrightarrow x!i\ -e\rightarrow\ x!Suc\ i) \longrightarrow snd(x!k){\in}p \wedge fst(x!j){=}fst(x!k)$
⟨*proof*⟩

### 3.5.1 Soundness of the System for Component Programs

#### Soundness of the Basic rule

**lemma** *unique-ctran-Basic* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$

⟨*proof*⟩

**lemma** *exists-ctran-Basic-None* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

**lemma** *Basic-sound*:
$[\![pre \subseteq \{s.\ f\ s \in post\};\ \{(s,\ t).\ s \in pre \wedge t = f\ s\} \subseteq guar;$
*stable pre rely*; *stable post rely*$]\!]$
$\Longrightarrow \models Basic\ f\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

### Soundness of the Await rule

**lemma** *unique-ctran-Await* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

**lemma** *exists-ctran-Await-None* [*rule-format*]:
$\forall s\ i.\ \ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

**lemma** *Star-imp-cptn*:
$(P,\ s)\ -c{*}\rightarrow\ (R,\ t) \Longrightarrow \exists l \in cp\ P\ s.\ (last\ l){=}(R,\ t)$
$\wedge\ (\forall i.\ Suc\ i{<}length\ l \longrightarrow l!i\ -c\rightarrow\ l!Suc\ i)$
⟨*proof*⟩

**lemma** *Await-sound*:
$[\![stable\ pre\ rely;\ stable\ post\ rely;$
$\forall V.\ \vdash P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post] \wedge$
$\models P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post]\ ]\!]$
$\Longrightarrow \models Await\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

### Soundness of the Conditional rule

**lemma** *Cond-sound*:
$[\![\ stable\ pre\ rely;\ \models P1\ sat\ [pre \cap b,\ rely,\ guar,\ post];$
$\models P2\ sat\ [pre \cap -\ b,\ rely,\ guar,\ post];\ \forall s.\ (s,s){\in}guar]\!]$
$\Longrightarrow \models (Cond\ b\ P1\ P2)\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

### Soundness of the Sequential rule

**inductive-cases** *Seq-cases* [*elim*!]: (*Some* (*Seq P Q*), *s*) $-c\rightarrow$ *t*

**lemma** *last-lift-not-None*: *fst* ((*lift Q*) ((*x*#*xs*)!(*length xs*))) $\neq$ *None*
⟨*proof*⟩

**lemma** *Seq-sound1* [*rule-format*]:
*x*∈ *cptn-mod* $\Longrightarrow$ ∀ *s P*. *x* !*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$
(∀ *i*<*length x*. *fst*(*x*!*i*)≠*Some Q*) $\longrightarrow$
(∃ *xs*∈ *cp* (*Some P*) *s*. *x*=*map* (*lift Q*) *xs*)
⟨*proof*⟩

**lemma** *Seq-sound2* [*rule-format*]:
*x* ∈ *cptn* $\Longrightarrow$ ∀ *s P i*. *x*!*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$ *i*<*length x*
$\longrightarrow$ *fst*(*x*!*i*)=*Some Q* $\longrightarrow$
(∀ *j*<*i*. *fst*(*x*!*j*)≠(*Some Q*)) $\longrightarrow$
(∃ *xs ys*. *xs* ∈ *cp* (*Some P*) *s* ∧ *length xs*=*Suc i*
∧ *ys* ∈ *cp* (*Some Q*) (*snd*(*xs* !*i*)) ∧ *x*=(*map* (*lift Q*) *xs*)@*tl ys*)
⟨*proof*⟩

**lemma** *last-lift-not-None2*: *fst* ((*lift Q*) (*last* (*x*#*xs*))) $\neq$ *None*
⟨*proof*⟩

**lemma** *Seq-sound*:
⟦⊨ *P sat* [*pre*, *rely*, *guar*, *mid*]; ⊨ *Q sat* [*mid*, *rely*, *guar*, *post*]⟧
$\Longrightarrow$ ⊨ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

### Soundness of the While rule

**lemma** *last-append*[*rule-format*]:
∀ *xs*. *ys*≠[] $\longrightarrow$ ((*xs*@*ys*)!(*length* (*xs*@*ys*) − (*Suc 0*)))=(*ys*!(*length ys* − (*Suc 0*)))
⟨*proof*⟩

**lemma** *assum-after-body*:
⟦ ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*];
(*Some P*, *s*) # *xs* ∈ *cptn-mod*; *fst* (*last* ((*Some P*, *s*) # *xs*)) = *None*; *s* ∈ *b*;
(*Some* (*While b P*), *s*) # (*Some* (*Seq P* (*While b P*)), *s*) #
*map* (*lift* (*While b P*)) *xs* @ *ys* ∈ *assum* (*pre*, *rely*)⟧
$\Longrightarrow$ (*Some* (*While b P*), *snd* (*last* ((*Some P*, *s*) # *xs*))) # *ys* ∈ *assum* (*pre*, *rely*)
⟨*proof*⟩

**lemma** *While-sound-aux* [*rule-format*]:
⟦ *pre* ∩ − *b* ⊆ *post*; ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*, *s*) ∈ *guar*;
*stable pre rely*; *stable post rely*; *x* ∈ *cptn-mod* ⟧
$\Longrightarrow$ ∀ *s xs*. *x*=(*Some*(*While b P*),*s*)#*xs* $\longrightarrow$ *x*∈*assum*(*pre*, *rely*) $\longrightarrow$ *x* ∈ *comm*
(*guar*, *post*)
⟨*proof*⟩

**lemma** *While-sound*:
⟦*stable pre rely*; *pre* ∩ − *b* ⊆ *post*; *stable post rely*;
⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*,*s*)∈*guar*⟧
⟹ ⊨ *While b P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

### Soundness of the Rule of Consequence

**lemma** *Conseq-sound*:
⟦*pre* ⊆ *pre*′; *rely* ⊆ *rely*′; *guar*′ ⊆ *guar*; *post*′ ⊆ *post*;
⊨ *P sat* [*pre*′, *rely*′, *guar*′, *post*′]⟧
⟹ ⊨ *P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

### Soundness of the system for sequential component programs

**theorem** *rgsound*:
⊢ *P sat* [*pre*, *rely*, *guar*, *post*] ⟹ ⊨ *P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

## 3.5.2 Soundness of the System for Parallel Programs

**definition** *ParallelCom* :: (′*a rgformula*) *list* ⇒ ′*a par-com* **where**
*ParallelCom Ps* ≡ *map* (*Some* ∘ *fst*) *Ps*

**lemma** *two*:
⟦ ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
*pre* ⊆ (⋂ *i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i*<*length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
*length xs*=*length clist*; *x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x*∈*par-assum*(*pre*, *rely*);
∀ *i*<*length clist*. *clist*!*i*∈*cp* (*Some*(*Com*(*xs*!*i*))) *s*; *x* ∝ *clist* ⟧
⟹ ∀ *j i*. *i*<*length clist* ∧ *Suc j*<*length x* ⟶ (*clist*!*i*!*j*) −*c*→ (*clist*!*i*!*Suc j*)
⟶ (*snd*(*clist*!*i*!*j*), *snd*(*clist*!*i*!*Suc j*)) ∈ *Guar*(*xs*!*i*)
⟨*proof*⟩

**lemma** *three* [*rule-format*]:
⟦ *xs*≠[]; ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
*pre* ⊆ (⋂ *i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i*<*length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
*length xs*=*length clist*; *x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum*(*pre*, *rely*);
∀ *i*<*length clist*. *clist*!*i*∈*cp* (*Some*(*Com*(*xs*!*i*))) *s*; *x* ∝ *clist* ⟧
⟹ ∀ *j i*. *i*<*length clist* ∧ *Suc j*<*length x* ⟶ (*clist*!*i*!*j*) −*e*→ (*clist*!*i*!*Suc j*)
⟶ (*snd*(*clist*!*i*!*j*), *snd*(*clist*!*i*!*Suc j*)) ∈ *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}.
*Guar* (*xs* ! *j*))

⟨*proof*⟩

**lemma** *four*:
⟦*xs*≠[]; ∀ *i* < *length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
(⋃*j*∈{*j*. *j* < *length xs*}. *Guar* (*xs* ! *j*)) ⊆ *guar*;
*pre* ⊆ (⋂*i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i* < *length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
*x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum* (*pre*, *rely*); *Suc i* < *length x*;
*x* ! *i* −*pc*→ *x* ! *Suc i*⟧
⟹ (*snd* (*x* ! *i*), *snd* (*x* ! *Suc i*)) ∈ *guar*
⟨*proof*⟩

**lemma** *parcptn-not-empty* [*simp*]:[] ∉ *par-cptn*
⟨*proof*⟩

**lemma** *five*:
⟦*xs*≠[]; ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
*pre* ⊆ (⋂*i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
(⋂*i*∈{*i*. *i* < *length xs*}. *Post* (*xs* ! *i*)) ⊆ *post*;
∀ *i* < *length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
*x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum* (*pre*, *rely*);
*All-None* (*fst* (*last x*)) ⟧ ⟹ *snd* (*last x*) ∈ *post*
⟨*proof*⟩

**lemma** *ParallelEmpty* [*rule-format*]:
∀ *i s*. *x* ∈ *par-cp* (*ParallelCom* []) *s* ⟶
*Suc i* < *length x* ⟶ (*x* ! *i*, *x* ! *Suc i*) ∉ *par-ctran*
⟨*proof*⟩

**theorem** *par-rgsound*:
⊢ *c SAT* [*pre*, *rely*, *guar*, *post*] ⟹
⊨ (*ParallelCom c*) *SAT* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

**end**

## 3.6 Concrete Syntax

**theory** *RG-Syntax*
**imports** *RG-Hoare Quote-Antiquote*
**begin**

**abbreviation** *Skip* :: *'a com* (‹*SKIP*›)
**where** *SKIP* ≡ *Basic id*

**notation** *Seq* (‹(-;;/ -)› [60,61] 60)

**syntax**
-Assign :: *idt* ⇒ *'b* ⇒ *'a com* (‹(´- :=/ -)› [70, 65] 61)
-Cond :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ⇒ *'a com* (‹(0IF -/ THEN -/ ELSE -/FI)› [0, 0, 0] 61)
-Cond2 :: *'a bexp* ⇒ *'a com* ⇒ *'a com* (‹(0IF - THEN - FI)› [0,0] 56)
-While :: *'a bexp* ⇒ *'a com* ⇒ *'a com* (‹(0WHILE - /DO - /OD)› [0, 0] 61)
-Await :: *'a bexp* ⇒ *'a com* ⇒ *'a com* (‹(0AWAIT - /THEN /- /END)› [0,0] 61)
-Atom :: *'a com* ⇒ *'a com* (‹(⟨-⟩)› 61)
-Wait :: *'a bexp* ⇒ *'a com* (‹(0WAIT - END)› 61)

**translations**
´*x* := *a* ⇀ *CONST Basic* «´(-*update-name x* (λ-. *a*))»
*IF b THEN c1 ELSE c2 FI* ⇀ *CONST Cond* {|*b*|} *c1 c2*
*IF b THEN c FI* ⇌ *IF b THEN c ELSE SKIP FI*
*WHILE b DO c OD* ⇀ *CONST While* {|*b*|} *c*
*AWAIT b THEN c END* ⇌ *CONST Await* {|*b*|} *c*
⟨*c*⟩ ⇌ *AWAIT CONST True THEN c END*
*WAIT b END* ⇌ *AWAIT b THEN SKIP END*

**nonterminal** *prgs*

**syntax**
-PAR :: *prgs* ⇒ *'a* (‹*COBEGIN//-//COEND*› 60)
-prg :: *'a* ⇒ *prgs* (‹-› 57)
-prgs :: [*'a*, *prgs*] ⇒ *prgs* (‹-//‖//-› [60,57] 57)

**translations**
-*prg a* ⇀ [*a*]
-*prgs a ps* ⇀ *a* # *ps*
-*PAR ps* ⇀ *ps*

**syntax**
-*prg-scheme* :: [*'a*, *'a*, *'a*, *'a*] ⇒ *prgs* (‹*SCHEME* [- ≤ - < -] -› [0,0,0,60] 57)

**translations**
-*prg-scheme j i k c* ⇌ (*CONST map* (λ*i*. *c*) [*j*..<*k*])

Translations for variables before and after a transition:

**syntax**
-*before* :: *id* ⇒ *'a* (‹$^{o}$-›)
-*after* :: *id* ⇒ *'a* (‹$^{a}$-›)

**translations**
$^{o}x$ ⇌ *x* ´ *CONST fst*

$^{a}x \rightleftharpoons x$ ´*CONST snd*

⟨*ML*⟩

**end**

## 3.7 Examples

**theory** *RG-Examples*
**imports** *RG-Syntax*
**begin**

**lemmas** *definitions* [*simp*]= *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

### 3.7.1 Set Elements of an Array to Zero

**lemma** *le-less-trans2*: ⟦(*j*::*nat*)<*k*; *i*≤ *j*⟧ ⟹ *i*<*k*
⟨*proof*⟩

**lemma** *add-le-less-mono*: ⟦ (*a*::*nat*) < *c*; *b*≤*d* ⟧ ⟹ *a* + *b* < *c* + *d*
⟨*proof*⟩

**record** *Example1* =
*A* :: *nat list*

**lemma** *Example1*:
⊢ *COBEGIN*
*SCHEME* [*0* ≤ *i* < *n*]
(´*A* := ´*A* [*i* := *0*],
{⦇ *n* < *length* ´*A* ⦈},
{⦇ *length* $^{o}A$ = *length* $^{a}A$ ∧ $^{o}A$ ! *i* = $^{a}A$ ! *i* ⦈},
{⦇ *length* $^{o}A$ = *length* $^{a}A$ ∧ (∀*j*<*n*. *i* ≠ *j* ⟶ $^{o}A$ ! *j* = $^{a}A$ ! *j*) ⦈},
{⦇ ´*A* ! *i* = *0* ⦈)
*COEND*
*SAT* [{⦇ *n* < *length* ´*A* ⦈, {⦇ $^{o}A$ = $^{a}A$ ⦈, {⦇ *True* ⦈, {⦇ ∀ *i* < *n*. ´*A* ! *i* = *0* ⦈]
⟨*proof*⟩

**lemma** *Example1-parameterized*:
*k* < *t* ⟹
⊢ *COBEGIN*
*SCHEME* [*k*∗*n*≤*i*<(*Suc k*)∗*n*] (´*A*:=´*A*[*i*:=*0*],
{⦇*t*∗*n* < *length* ´*A*⦈},
{⦇*t*∗*n* < *length* $^{o}A$ ∧ *length* $^{o}A$=*length* $^{a}A$ ∧ $^{o}A$!*i* = $^{a}A$!*i*⦈},
{⦇*t*∗*n* < *length* $^{o}A$ ∧ *length* $^{o}A$=*length* $^{a}A$ ∧ (∀*j*<*length* $^{o}A$ . *i*≠*j* ⟶ $^{o}A$!*j* = $^{a}A$!*j*)⦈},
{⦇´*A*!*i*=*0*⦈)
*COEND*
*SAT* [{⦇*t*∗*n* < *length* ´*A*⦈},
{⦇*t*∗*n* < *length* $^{o}A$ ∧ *length* $^{o}A$=*length* $^{a}A$ ∧ (∀ *i*<*n*. $^{o}A$!(*k*∗*n*+*i*)=$^{a}A$!(*k*∗*n*+*i*))⦈},

$\{\!|t{*}n < \mathit{length}\ {}^{o}A \wedge \mathit{length}\ {}^{o}A{=}\mathit{length}\ {}^{a}A \wedge$
$(\forall\, i{<}\mathit{length}\ {}^{o}A\ .\ (i{<}k{*}n \longrightarrow {}^{o}A!i = {}^{a}A!i) \wedge ((\mathit{Suc}\ k){*}n \leq i \longrightarrow {}^{o}A!i = {}^{a}A!i))|\!\},$
$\{\!|\forall\, i{<}n.\ {}^{\prime}A!(k{*}n{+}i) = 0|\!\}]$
$\langle \mathit{proof} \rangle$

### 3.7.2 Increment a Variable in Parallel

#### Two components

**record** *Example2* =
*x* :: *nat*
*c-0* :: *nat*
*c-1* :: *nat*

**lemma** *Example2*:
$\vdash$ *COBEGIN*
$(\langle\ {}^{\prime}x{:=}{}^{\prime}x{+}1;;\ {}^{\prime}c\text{-}0{:=}{}^{\prime}c\text{-}0\ +\ 1\ \rangle,$
$\{\!|{}^{\prime}x{=}{}^{\prime}c\text{-}0\ +\ {}^{\prime}c\text{-}1\ \wedge\ {}^{\prime}c\text{-}0{=}0|\!\},$
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{\prime}x{=}{}^{\prime}c\text{-}0 + {}^{\prime}c\text{-}1 \wedge {}^{\prime}c\text{-}0{=}1\ |\!\})$
$\|$
$(\langle\ {}^{\prime}x{:=}{}^{\prime}x{+}1;;\ {}^{\prime}c\text{-}1{:=}{}^{\prime}c\text{-}1{+}1\ \rangle,$
$\{\!|{}^{\prime}x{=}{}^{\prime}c\text{-}0\ +\ {}^{\prime}c\text{-}1 \wedge {}^{\prime}c\text{-}1{=}0\ |\!\},$
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{\prime}x{=}{}^{\prime}c\text{-}0 + {}^{\prime}c\text{-}1 \wedge {}^{\prime}c\text{-}1{=}1|\!\})$
*COEND*
*SAT* $[\{\!|{}^{\prime}x{=}0 \wedge {}^{\prime}c\text{-}0{=}0 \wedge {}^{\prime}c\text{-}1{=}0|\!\},$
$\{\!|{}^{o}x{=}{}^{a}x \wedge\ {}^{o}c\text{-}0{=}\ {}^{a}c\text{-}0 \wedge {}^{o}c\text{-}1{=}{}^{a}c\text{-}1|\!\},$
$\{\!|\mathit{True}|\!\},$
$\{\!|{}^{\prime}x{=}2|\!\}]$
$\langle \mathit{proof} \rangle$

#### Parameterized

**lemma** *Example2-lemma2-aux*: $j{<}n \Longrightarrow$
$(\sum i{=}0..{<}n.\ (b\ i{::}\mathit{nat})) =$
$(\sum i{=}0..{<}j.\ b\ i)\ +\ b\ j\ +\ (\sum i{=}0..{<}n{-}(\mathit{Suc}\ j)\ .\ b\ (\mathit{Suc}\ j\ +\ i))$
$\langle \mathit{proof} \rangle$

**lemma** *Example2-lemma2-aux2*:

$j \leq s \Longrightarrow (\sum i{::}nat{=}0..{<}j.\ (b\ (s{:=}t))\ i) = (\sum i{=}0..{<}j.\ b\ i)$
⟨*proof*⟩

**lemma** *Example2-lemma2*:
$[\![j{<}n;\ b\ j{=}0]\!] \Longrightarrow Suc\ (\sum i{::}nat{=}0..{<}n.\ b\ i){=}(\sum i{=}0..{<}n.\ (b\ (j := Suc\ 0))\ i)$
⟨*proof*⟩

**lemma** *Example2-lemma2-Suc0*: $[\![j{<}n;\ b\ j{=}0]\!] \Longrightarrow$
$Suc\ (\sum i{::}nat{=}0..{<}\ n.\ b\ i){=}(\sum i{=}0..{<}\ n.\ (b\ (j{:=}Suc\ 0))\ i)$
⟨*proof*⟩

**record** *Example2-parameterized* =
$C :: nat \Rightarrow nat$
$y :: nat$

**lemma** *Example2-parameterized*: $0{<}n \Longrightarrow$
$\vdash$ *COBEGIN SCHEME* $[0{\leq}i{<}n]$
$(\langle\ {}'y{:=}{}'y{+}1;;\ {}'C{:=}{}'C\ (i{:=}1)\ \rangle,$
$\{\!|{}'y{=}(\sum i{=}0..{<}n.\ {}'C\ i) \wedge {}'C\ i{=}0|\!\},$
$\{\!|{}^{o}C\ i = {}^{a}C\ i \wedge$
$({}^{o}y{=}(\sum i{=}0..{<}n.\ {}^{o}C\ i) \longrightarrow {}^{a}y\ {=}(\sum i{=}0..{<}n.\ {}^{a}C\ i))|\!\},$
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}C\ j = {}^{a}C\ j) \wedge$
$({}^{o}y{=}(\sum i{=}0..{<}n.\ {}^{o}C\ i) \longrightarrow {}^{a}y\ {=}(\sum i{=}0..{<}n.\ {}^{a}C\ i))|\!\},$
$\{\!|{}'y{=}(\sum i{=}0..{<}n.\ {}'C\ i) \wedge {}'C\ i{=}1|\!\})$
*COEND*
*SAT* $[\{\!|{}'y{=}0 \wedge (\sum i{=}0..{<}n.\ {}'C\ i){=}0\ |\!\},\ \{\!|{}^{o}C{=}{}^{a}C \wedge {}^{o}y{=}{}^{a}y|\!\},\ \{\!|True|\!\},\ \{\!|{}'y{=}n|\!\}]$
⟨*proof*⟩

### 3.7.3 Find Least Element

A previous lemma:

**lemma** *mod-aux* :$[\![i < (n{::}nat);\ a\ mod\ n = i;\ \ j < a + n;\ j\ mod\ n = i;\ a < j]\!] \Longrightarrow$
*False*
⟨*proof*⟩

**record** *Example3* =
$X :: nat \Rightarrow nat$
$Y :: nat \Rightarrow nat$

**lemma** *Example3*: $m\ mod\ n{=}0 \Longrightarrow$
$\vdash$ *COBEGIN*
*SCHEME* $[0{\leq}i{<}n]$
(*WHILE* $(\forall j{<}n.\ {}'X\ i < {}'Y\ j)$ *DO*
*IF* $P(B!({}'X\ i))$ *THEN* ${}'Y{:=}{}'Y\ (i{:=}{}'X\ i)$
*ELSE* ${}'X{:=}\ {}'X\ (i{:=}({}'X\ i){+}\ n)$ *FI*
*OD*,
$\{\!|({}'X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}'X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}'Y\ i{<}m \longrightarrow P(B!({}'Y\ i)) \wedge {}'Y\ i{\leq}\ m{+}i)|\!\},$
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{a}Y\ j \leq {}^{o}Y\ j) \wedge {}^{o}X\ i = {}^{a}X\ i \wedge$

$^{o}Y\ i = {}^{a}Y\ i\}\!\rangle$,
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}X\ j = {}^{a}X\ j \wedge {}^{o}Y\ j = {}^{a}Y\ j) \wedge$
$^{a}Y\ i \leq {}^{o}Y\ i\}\!\rangle$,
$\{\!|(\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y\ i{<}m \longrightarrow P(B!(\acute{}Y\ i)) \wedge \acute{}Y\ i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y\ j \leq \acute{}X\ i)\ \}\!\rangle$
*COEND*
*SAT* $[\{\!|\ \forall i{<}n.\ \acute{}X\ i{=}i \wedge \acute{}Y\ i{=}m{+}i\ \}\!\rangle, \{\!|{}^{o}X{=}{}^{a}X \wedge {}^{o}Y{=}{}^{a}Y\}\!\rangle, \{\!|True\}\!\rangle,$
$\{\!|\forall i{<}n.\ (\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$(\acute{}Y\ i{<}m \longrightarrow P(B!(\acute{}Y\ i)) \wedge \acute{}Y\ i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y\ j \leq \acute{}X\ i)\}\!\rangle]$
⟨*proof*⟩

Same but with a list as auxiliary variable:

**record** *Example3-list* =
$X$ :: *nat list*
$Y$ :: *nat list*

**lemma** *Example3-list*: $m\ mod\ n{=}0 \Longrightarrow \vdash$ (*COBEGIN SCHEME* $[0{\leq}i{<}n]$
(*WHILE* $(\forall j{<}n.\ \acute{}X!i < \acute{}Y!j)$ *DO*
*IF* $P(B!(\acute{}X!i))$ *THEN* $\acute{}Y{:=}\acute{}Y[i{:=}\acute{}X!i]$ *ELSE* $\acute{}X{:=}\ \acute{}X[i{:=}(\acute{}X!i){+}\ n]$ *FI*
*OD*,
$\{\!|n{<}length\ \acute{}X \wedge n{<}length\ \acute{}Y \wedge (\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i)\}\!\rangle$,
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{a}Y!j \leq {}^{o}Y!j) \wedge {}^{o}X!i = {}^{a}X!i \wedge$
$^{o}Y!i = {}^{a}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y\}\!\rangle$,
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}X!j = {}^{a}X!j \wedge {}^{o}Y!j = {}^{a}Y!j) \wedge$
$^{a}Y!i \leq {}^{o}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y\}\!\rangle$,
$\{\!|(\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y!j \leq \acute{}X!i)\ \}\!\rangle$ *COEND*)
*SAT* $[\{\!|n{<}length\ \acute{}X \wedge n{<}length\ \acute{}Y \wedge (\forall i{<}n.\ \acute{}X!i{=}i \wedge \acute{}Y!i{=}m{+}i)\ \}\!\rangle,$
$\{\!|{}^{o}X{=}{}^{a}X \wedge {}^{o}Y{=}{}^{a}Y\}\!\rangle,$
$\{\!|\ True\}\!\rangle,$
$\{\!|\forall i{<}n.\ (\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$(\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y!j \leq \acute{}X!i)\}\!\rangle]$
⟨*proof*⟩

**end**
**theory** *Hoare-Parallel*
**imports** *OG-Examples Gar-Coll Mul-Gar-Coll RG-Examples*
**begin**

**end**

# Bibliography

[1] Leonor Prensa Nieto. *Verification of Parallel Programs with the Owicki-Gries and Rely-Guarantee Methods in Isabelle/HOL.* PhD thesis, Technische Universität München, 2002.

[2] Tobias Nipkow and Leonor Prensa Nieto. Owicki/Gries in Isabelle/HOL. In J.-P. Finance, editor, *Fundamental Approaches to Software Engineering (FASE'99)*, volume 1577 of *LNCS*, pages 188–203. Springer, 1999.

[3] Leonor Prensa Nieto. The Rely-Guarantee method in Isabelle/HOL. In P. Degano, editor, *European Symposium on Programming (ESOP'03)*, volume 2618, pages 348–362, 2003.

[4] Leonor Prensa Nieto and Javier Esparza. Verifying single and multi-mutator garbage collectors with Owicki/Gries in Isabelle/HOL. In M. Nielsen and B. Rovan, editors, *Mathematical Foundations of Computer Science (MFCS 2000)*, volume 1893 of *LNCS*, pages 619–628. Springer-Verlag, 2000.