Homology

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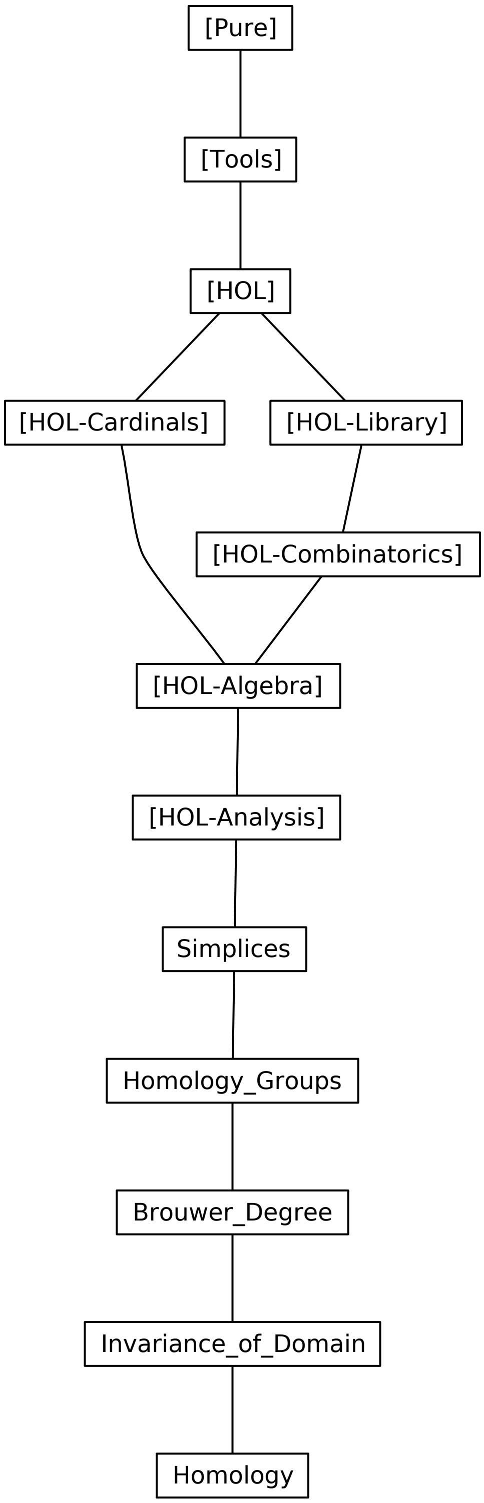
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0.1 Homology, I: Simplices

theory *Simplices*
 imports
  HOL−Analysis.Function_Metric
  HOL−Analysis.Abstract_Euclidean_Space
  HOL−Algebra.Free_Abelian_Groups
begin

0.1.1 Standard simplices, all of which are topological subspaces of R^n.

0.1.2 Face map

0.1.3 Singular simplices, forcing canonicity outside the intended domain

0.1.4 Singular chains

0.1.5 Boundary homomorphism for singular chains

0.1.6 Factoring out chains in a subtopology for relative homology

0.1.7 Relative cycles $Z_pX(S)$ where X is a topology and S a subset

0.1.8 Relative boundaries B_pXS, where X is a topology and S a subset.

0.1.9 The (relative) homology relation

0.1.10 Show that all boundaries are cycles, the key "chain complex" property.

0.1.11 Operations induced by a continuous map g between topological spaces

0.1.12 Homology of one-point spaces degenerates except for $p = 0$.

0.1.13 Simplicial chains

0.1.14 The cone construction on simplicial simplices.

0.1.15 Barycentric subdivision of a linear ("simplicial") simplex's image

0.1.16 Singular subdivision

0.1.17 Excision argument that we keep doing singular subdivision

proposition *sufficient_iterated_singular_subdivision_exists*:
assumes $\mathcal{C}$: $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *openin* $X\ U$
and X: *topspace* $X \subseteq \bigcup\mathcal{C}$
and p: *singular_chain* $p\ X\ c$
obtains n **where** $\bigwedge m\ f.\ [\![n \leq m;\ f \in$ *Poly_Mapping.keys* ((*singular_subdivision* p ^^ m) c)$]\!]$
$\Longrightarrow \exists\ V \in \mathcal{C}.\ f \in$ (*standard_simplex* p) $\rightarrow V$

0.1.18 Homotopy invariance

theorem *homotopic_imp_homologous_rel_chain_maps*:
assumes *hom*: *homotopic_with* ($\lambda h.\ h \in T \rightarrow V$) $S\ U\ f\ g$ **and** c: *singular_relcycle* $p\ S\ T\ c$
shows *homologous_rel* $p\ U\ V$ (*chain_map* $p\ f\ c$) (*chain_map* $p\ g\ c$)

end

0.2 Homology, II: Homology Groups

theory *Homology_Groups*
 imports *Simplices HOL−Algebra.Exact_Sequence*

begin

0.2.1 Homology Groups

0.2.2 Towards the Eilenberg-Steenrod axioms

proposition *homology_homotopy_axiom*:
 assumes *homotopic_with* $(\lambda h.\ h \in S \to T)\ X\ Y\ f\ g$
 shows *hom_induced* $p\ X\ S\ Y\ T\ f =$ *hom_induced* $p\ X\ S\ Y\ T\ g$

proposition *homology_excision_axiom*:
 assumes X *closure_of* $U \subseteq X$ *interior_of* $T\ T \subseteq S$
 shows
 hom_induced p (*subtopology* X $(S - U)$) $(T - U)$ (*subtopology* X S) T *id*
 $\in$ *iso* (*relative_homology_group* p (*subtopology* X $(S - U)$) $(T - U)$)
 (*relative_homology_group* p (*subtopology* X S) T)

0.2.3 Additivity axiom

proposition *iso_cycle_group_sum*:
 assumes *disj*: *pairwise disjnt* $\mathcal{U}$ **and** *UU*: $\bigcup\mathcal{U} =$ *topspace* X

and *subs*: $\bigwedge C\ T.\ [\![$*compactin* X C; *path_connectedin* X C; $T \in \mathcal{U}$; $\neg$ *disjnt* C $T]\!] \Longrightarrow C \subseteq T$
shows ($\lambda f.$ *sum′* f $\mathcal{U}$) $\in$ *iso* (*sum_group* $\mathcal{U}$ ($\lambda T.$ *relcycle_group* p (*subtopology* X T) {}))
(*relcycle_group* p X {})

proposition *homology_additivity_axiom_gen*:
assumes *disj*: *pairwise disjnt* $\mathcal{U}$ **and** *UU*: $\bigcup\mathcal{U}$ = *topspace* X
and *subs*: $\bigwedge C\ T.\ [\![$*compactin* X C; *path_connectedin* X C; $T \in \mathcal{U}$; $\neg$ *disjnt* C $T]\!] \Longrightarrow C \subseteq T$
shows ($\lambda x.$ *gfinprod* (*homology_group* p X)
($\lambda V.$ *hom_induced* p (*subtopology* X V) {} X {} *id* (x V)) $\mathcal{U}$)
$\in$ *iso* (*sum_group* $\mathcal{U}$ ($\lambda S.$ *homology_group* p (*subtopology* X S))) (*homology_group* p X)
(**is** *?h* $\in$ *iso* *?SG* *?HG*)

corollary *homology_additivity_axiom*:
assumes *disj*: *pairwise disjnt* $\mathcal{U}$ **and** *UU*: $\bigcup\mathcal{U}$ = *topspace* X
and *ope*: $\bigwedge v.\ v \in \mathcal{U} \Longrightarrow$ *openin* X v
shows ($\lambda x.$ *gfinprod* (*homology_group* p X)
($\lambda v.$ *hom_induced* p (*subtopology* X v) {} X {} *id* (x v)) $\mathcal{U}$)
$\in$ *iso* (*sum_group* $\mathcal{U}$ ($\lambda S.$ *homology_group* p (*subtopology* X S))) (*homology_group* p X)

0.2.4 Special properties of singular homology

proposition *iso_integer_zeroth_homology_group*:
assumes X: *path_connected_space* X **and** f: *singular_simplex 0* X f
shows *pow* (*homology_group 0* X) (*homologous_rel_set 0* X {} (*frag_of* f))
$\in$ *iso integer_group* (*homology_group 0* X) (**is** *pow ?H ?q* $\in$ *iso _ ?H*)

corollary *isomorphic_integer_zeroth_homology_group*:
assumes X: *path_connected_space* X *topspace* $X \neq$ {}
shows *homology_group 0* $X \cong$ *integer_group*

corollary *homology_coefficients*:
topspace $X = \{a\} \Longrightarrow$ *homology_group 0* $X \cong$ *integer_group*

proposition *zeroth_homology_group*:
homology_group 0 $X \cong$ *free_Abelian_group* (*path_components_of* X)

0.2.5 More basic properties of homology groups, deduced from the E-S axioms

corollary *mon_hom_induced_section_map*:
  assumes *section_map X Y f*
  shows (*hom_induced p X* {} *Y* {} *f*) ∈ *mon* (*homology_group p X*) (*homology_group p Y*)

corollary *epi_hom_induced_retraction_map*:
  assumes *retraction_map X Y f*
  shows (*hom_induced p X* {} *Y* {} *f*) ∈ *epi* (*homology_group p X*) (*homology_group p Y*)

0.2.6 Generalize exact homology sequence to triples

proposition *homology_exactness_triple_1*:
  assumes $T \subseteq S$
  shows *exact_seq* ([*relative_homology_group*(*p*−*1*) (*subtopology X S*) *T*,
        relative_homology_group p X S,
        relative_homology_group p X T],
       [*hom_relboundary p X S T*, *hom_induced p X T X S id*])
    (**is** *exact_seq* ([*?G1*,*?G2*,*?G3*], [*?h1*,*?h2*]))

proposition *homology_exactness_triple_2*:
  assumes $T \subseteq S$
  shows *exact_seq* ([*relative_homology_group*(*p*−*1*) *X T*,
        relative_homology_group(*p*−*1*) (*subtopology X S*) *T*,
        relative_homology_group p X S],
       [*hom_induced* (*p*−*1*) (*subtopology X S*) *T X T id*, *hom_relboundary p X S T*])
    (**is** *exact_seq* ([*?G1*,*?G2*,*?G3*], [*?h1*,*?h2*]))

proposition *homology_exactness_triple_3*:
  assumes $T \subseteq S$

shows *exact_seq* ([*relative_homology_group p X S*,
relative_homology_group p X T,
relative_homology_group p (*subtopology X S*) *T*],
[*hom_induced p X T X S id*, *hom_induced p* (*subtopology X S*) *T X T id*])
(**is** *exact_seq* ([*?G1*,*?G2*,*?G3*], [*?h1*,*?h2*]))

end

0.3 Homology, III: Brouwer Degree

theory *Brouwer_Degree*
imports *Homology_Groups HOL−Algebra.Multiplicative_Group*

begin

0.3.1 Reduced Homology

0.3.2 More homology properties of deformations, retracts, contractible spaces

corollary *isomorphic_relative_homology_groups_relativization_contractible*:
assumes *contractible_space*(*subtopology X S*) *contractible_space*(*subtopology X T*) $T \subseteq S$ *topspace* $X \cap T \neq \{\}$
shows *relative_homology_group p X T* $\cong$ *relative_homology_group p X S*

corollary *isomorphic_relative_homology_groups_inclusion_contractible*:
assumes *contractible_space X contractible_space*(*subtopology X S*) $T \subseteq S$ *topspace* $X \cap S \neq \{\}$

shows *relative_homology_group p (subtopology X S) T* ≅ *relative_homology_group p X T*

corollary *isomorphic_relative_homology_groups_relboundary_contractible*:
assumes *contractible_space X contractible_space(subtopology X T)* $T \subseteq S$ *topspace* $X \cap T \neq \{\}$
shows *relative_homology_group p X S* ≅ *relative_homology_group* $(p - 1)$ *(subtopology X S) T*

0.3.3 Homology groups of spheres

proposition *iso_relative_homology_group_upper_hemisphere*:
(*hom_induced p (subtopology (nsphere n)* $\{x.\ x\ k \geq 0\}$) $\{x.\ x\ k = 0\}$ *(nsphere n)* $\{x.\ x\ k \leq 0\}$ *id*)
∈ *iso* (*relative_homology_group p (subtopology (nsphere n)* $\{x.\ x\ k \geq 0\}$) $\{x.\ x\ k = 0\}$)
(*relative_homology_group p (nsphere n)* $\{x.\ x\ k \leq 0\}$) (**is** *?h* ∈ *iso ?G ?H*)

corollary *iso_upper_hemisphere_reduced_homology_group*:
(*hom_boundary* $(1 + p)$ *(subtopology (nsphere (Suc n))* $\{x.\ x(Suc\ n) \geq 0\}$) $\{x.\ x(Suc\ n) = 0\}$)
∈ *iso* (*relative_homology_group* $(1 + p)$ *(subtopology (nsphere (Suc n))* $\{x.\ x(Suc\ n) \geq 0\}$) $\{x.\ x(Suc\ n) = 0\}$)
(*reduced_homology_group p (nsphere n)*)

corollary *iso_reduced_homology_group_upper_hemisphere*:
assumes $k \leq n$
shows *hom_induced p (nsphere n)* {} *(nsphere n)* $\{x.\ x\ k \geq 0\}$ *id*
∈ *iso (reduced_homology_group p (nsphere n)) (relative_homology_group p (nsphere n)* $\{x.\ x\ k \geq 0\}$)

0.3.4 Brouwer degree of a Map

corollary *Brouwer_degree2_nonsurjective*:
$[\![$*continuous_map*(*nsphere p*) (*nsphere p*) *f*; *f* ' *topspace* (*nsphere p*) $\neq$ *topspace* (*nsphere p*)$]\!]$
$\Longrightarrow$ *Brouwer_degree2 p f* = *0*

proposition *Brouwer_degree2_reflection*:
Brouwer_degree2 p (λx *i*. *if i* = *0 then* $-x$ *i else x i*) = -1 (**is** *Brouwer_degree2 _ ?r* = -1)

end

0.4 Invariance of Domain

theory *Invariance_of_Domain*
imports *Brouwer_Degree HOL−Analysis.Continuous_Extension HOL−Analysis.Homeomorphism*

begin

0.4.1 Degree invariance mod 2 for map between pairs

theorem *Borsuk_odd_mapping_degree_step*:
assumes *cmf*: *continuous_map* (*nsphere n*) (*nsphere n*) *f*
and *f*: $\bigwedge u$. $u \in$ *topspace*(*nsphere n*) $\Longrightarrow$ ($f \circ (\lambda x\ i.\ -x\ i)$) u = (($\lambda x\ i.\ -x\ i$) $\circ$ f) u
and *fim*: $f \in$ (*topspace*(*nsphere*(n − *Suc 0*))) $\to$ *topspace*(*nsphere*(n − *Suc 0*))
shows *even* (*Brouwer_degree2 n f* − *Brouwer_degree2* (n − *Suc 0*) *f*)

0.4.2 General Jordan-Brouwer separation theorem and invariance of dimension

proposition *relative_homology_group_Euclidean_complement_step*:
assumes *closedin* (*Euclidean_space n*) *S*
shows *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S*)
$\cong$ *relative_homology_group* ($p + k$) (*Euclidean_space* (n+k)) (*topspace*(*Euclidean_space* (n+k)) − *S*)

proposition *isomorphic_relative_homology_groups_Euclidean_complements*:
assumes *S*: *closedin* (*Euclidean_space n*) *S* **and** *T*: *closedin* (*Euclidean_space n*) *T*
and *hom*: (*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S*)

≅ *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *T*)

theorem *invariance_of_dimension_Euclidean_space*:
Euclidean_space m homeomorphic_space Euclidean_space n ⟷ *m* = *n*

theorem *invariance_of_domain_Euclidean_space*:
assumes *U*: *openin* (*Euclidean_space n*) *U*
and *cmf*: *continuous_map* (*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *openin* (*Euclidean_space n*) (*f* ' *U*) (**is** *openin ?E* (*f* ' *U*))

corollary *invariance_of_domain_Euclidean_space_embedding_map*:
assumes *openin* (*Euclidean_space n*) *U*
and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *embedding_map*(*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*

corollary *invariance_of_domain_Euclidean_space_gen*:
assumes *n* ≤ *m* **and** *U*: *openin* (*Euclidean_space m*) *U*
and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *openin* (*Euclidean_space n*) (*f* ' *U*)

corollary *invariance_of_domain_Euclidean_space_embedding_map_gen*:
assumes *n* ≤ *m* **and** *U*: *openin* (*Euclidean_space m*) *U*
and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *embedding_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*

0.4.3 Relating two variants of Euclidean space, one within product topology.

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen_OLD*:
fixes B :: $'n$::*euclidean_space set*
assumes *finite B independent B* **and** *orth*: *pairwise orthogonal B* **and** *n*: *card B* $= n$
obtains *f g* **where** *homeomorphic_maps* (*Euclidean_space n*) (*top_of_set* (*span B*)) *f g*

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen*:
fixes B :: $'n$::*euclidean_space set*
assumes *independent B* **and** *orth*: *pairwise orthogonal B* **and** *n*: *card B* $= n$
and *1*: $\bigwedge u.\ u \in B \Longrightarrow norm\ u = 1$
obtains *f g* **where** *homeomorphic_maps* (*Euclidean_space n*) (*top_of_set* (*span B*)) *f g*
and $\bigwedge x.\ x \in topspace\ (Euclidean\_space\ n) \Longrightarrow (norm\ (f\ x))^2 = (\sum i{<}n.\ (x\ i)^2)$

corollary *homeomorphic_maps_Euclidean_space_euclidean*:
obtains f :: $(nat \Rightarrow real) \Rightarrow {'n}$::*euclidean_space* **and** *g*
where *homeomorphic_maps* (*Euclidean_space* (*DIM*($'n$))) *euclidean f g*

0.4.4 Invariance of dimension and domain

corollary *invariance_of_domain_subspaces*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V* **and** *VU*: $dim\ V \le dim\ U$
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \rightarrow V$
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (*f ‘ S*)

corollary *invariance_of_dimension_subspaces*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V*
and *contf*: *continuous_on S f* **and** *fim*: $f\ `\ S \subseteq V$
and *injf*: *inj_on f S* **and** $S \neq \{\}$
shows $dim\ U \le dim\ V$

corollary *invariance_of_domain_affine_sets*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V* $aff\_dim\ V \le aff\_dim\ U$
and *contf*: *continuous_on S f* **and** *fim*: $f\ `\ S \subseteq V$
and *injf*: *inj_on f S*

shows *openin* (*top_of_set V*) (*f* ‘ *S*)

corollary *invariance_of_dimension_affine_sets*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *aff_dim U* ≤ *aff_dim V*

corollary *invariance_of_dimension*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *contf*: *continuous_on S f* **and** *open S*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *DIM*(*'a*) ≤ *DIM*(*'b*)

corollary *continuous_injective_image_subspace_dim_le*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *subspace S subspace T*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *T*
and *injf*: *inj_on f S*
shows *dim S* ≤ *dim T*

corollary *invariance_of_domain_homeomorphic*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *open S continuous_on S f DIM*(*'b*) ≤ *DIM*(*'a*) *inj_on f S*
shows *S homeomorphic* (*f* ‘ *S*)

proposition *homeomorphic_interiors*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
assumes *S homeomorphic T interior S* = {} ⟷ *interior T* = {}
shows (*interior S*) *homeomorphic* (*interior T*)

proposition *uniformly_continuous_homeomorphism_UNIV_trivial*:
fixes *f* :: *'a::euclidean_space* ⇒ *'a*
assumes *contf*: *uniformly_continuous_on S f* **and** *hom*: *homeomorphism S UNIV f g*
shows *S* = *UNIV*

proposition *invariance_of_domain_sphere_affine_set_gen*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: *f* ‘ *S* ⊆ *T*
and *U*: *bounded U convex U*
and *affine T* **and** *affTU*: *aff_dim T* < *aff_dim U*
and *ope*: *openin* (*top_of_set* (*rel_frontier U*)) *S*
shows *openin* (*top_of_set T*) (*f* ‘ *S*)

end

theory *Homology*
imports *Invariance_of_Domain*
begin

end